Question

Determine two terms in the small ε expansion of the roots of

$$x^3 - 2x^2 - x + 2 = \varepsilon(3x^2 + 2x + 2)$$

Answer

$x^3 - 2x^2 - x + 2 = \varepsilon(3x^2 + 2x + 2)$

Not a singular perturbation as $\varepsilon \times O(x^2)$ and polynomial is 3rd order.

$\varepsilon = 0 \Rightarrow x^3 - 2x^2 - x + 2 = 0$ with roots $x = 1$ (obvious) by substitution, or $x^2 - x - 2 = 0 \Rightarrow x = 2$ or -1.

Therefore no degenerate roots at $\varepsilon = 0$

$\Rightarrow$ try ansatz $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$

where $x_0 = 1,\ -1$ pr 2. Find x_1 by substitution:

$$\left.\begin{array}{rcccl} x^3 & = & (x_0 + \varepsilon x_1) + O(\varepsilon^2))^3 & = & x_0^3 + 3\varepsilon x_1 x_0^2 + O(\varepsilon^2) \\ -2x^2 & = & -2(x - 0 + \varepsilon x_1 + O(\varepsilon^2))^2 & = & -2x_0^2 - 4\varepsilon x_0 x_1 + O(\varepsilon^2) \\ -x & = & -(x_0 + \varepsilon x_1 + O(\varepsilon^2)) & = & -x_0 - \varepsilon x_1 + O(\varepsilon^2) \\ +2 & = & +2 & = & +2 \end{array}\right\}$$

LHS$=x_0^3 - 2x_0^2 - x_0 + 2 + \varepsilon(3x_1x_0^2 - 4x_0x_1 - x_1) + O(\varepsilon^2)$

$$\left.\begin{array}{rcccl} 3\varepsilon x^2 & = & 3\varepsilon(x_0 + \varepsilon x_1 + O(\varepsilon^2))^2 & = & 3\varepsilon x_0^2 + O(\varepsilon^2) \\ 2\varepsilon x & = & 2\varepsilon(x_0 + \varepsilon x_1 + O(\varepsilon^2)) & = & 2\varepsilon x_0 + O(\varepsilon^2) \\ 2\varepsilon & = & 2\varepsilon & = & 2\varepsilon \end{array}\right\}$$

RHS$=\varepsilon(2 + 3x_0^2 + 2x_0) + O(\varepsilon^2)$

LHS=RHS $\Rightarrow x_0^3 - 2x_0^2 - x_0 + 2 + \varepsilon(3x_1x_0^2 - 4x_0x_1 - x_1 - 2 - 3x_0^2 - 2x_0) + O(\varepsilon^2) = 0$

Balancing:

$O(\varepsilon^0)$: Gives roots $x_0 = \pm$ and 2 as above

$O(\varepsilon^1) : 3x_1x_0^2 - 4x_0x_1 - x_1 - 2 - 3x_0^2 - 2x_0 = 0$

$$\text{so } x_0 = \left\{\begin{array}{rcl} +1 & \Rightarrow & x_1 = -\dfrac{7}{2} \\ -1 & \Rightarrow & x_1 = +\dfrac{1}{2} \\ 2 & \Rightarrow & x_1 = +6 \end{array}\right.$$

$$\text{Therefore } x = \left\{\begin{array}{l} 1 - \dfrac{7}{2}\varepsilon + O(\varepsilon^2) \\ -1 + \dfrac{1}{2}\varepsilon + O(\varepsilon^2) \\ 2 + \dfrac{1}{6}\varepsilon + O(\varepsilon^2) \end{array}\right.$$