Question

Bessel functions (of order v) are denoted by $J_v(x)$. They arise very frequently in the solution of wave problems with cylindrical symmetry. It is often required to know where the zero values of $J_v(x)$ are as a function o fx (being related to eigenvalues, energy levels or frequencies of vibration etc. or the problem in question).

The large x asymptotic expansion for $J_0(x)$ is given by,

$$J_0(x) \sim \sqrt{\frac{2}{\pi}} \left[\frac{\cos(x - \frac{\pi}{4})}{x^{\frac{1}{2}}} + \frac{\sin(x - \frac{\pi}{4})}{8x^{\frac{3}{2}}} \right] + O\left(\frac{1}{x^{\frac{5}{2}}}\right)$$

(i) Show that the roots as $x \to +\infty$ are given by $\cdots$

$$x \sim \left(n - \frac{1}{4}\right)\pi + \frac{1}{8(n + \frac{1}{4})\pi} + \cdots, \ n \to +\infty$$

(ii) Compare these with the first five numerically evaluate roots, and comment on this, given that n is a measure of the size of x.

Index	1	2	3	4	5
Root	2.40482...	5.52007...	8.65372...	11.79153...	14.93091...

Answer

(i) Clearly $J_0(x)$ is approximately zero (to $O(x^{-2})$)

when

$$\begin{aligned} \frac{\cos(x - \frac{\pi}{4})}{x^{\frac{1}{2}}} &= -\frac{\sin(x - \frac{\pi}{4})}{8x^{\frac{3}{2}}} \\ \Rightarrow \cot\left(x - \frac{\pi}{4}\right) &= -\frac{1}{8x} \quad (\star)(\star) \end{aligned}$$

so for $x \to \infty$

$\cot\left(x - \frac{\pi}{4}\right) = 0$ is a first approximation

$$\begin{aligned} -\frac{\pi}{4} + x &= \left(m + \frac{1}{2}\right)\pi \quad m \text{ integer} \\ x &= \left(m + \frac{3}{4}\right)\pi \end{aligned}$$

Calling $m = n - 1$ say we get (n another integer)

$$x = \left(n - \frac{1}{4}\right)\pi$$

Now to improve, let $n = \left(n - \dfrac{1}{4}\right)\pi + \delta,\ \delta = o(1)$

Substitute into $(\star)(\star)$

$\cot\left(\left(n - \dfrac{1}{2}\right)\pi + \delta\right) = -\dfrac{1}{8[(n-\frac{1}{4})\pi + \delta]}$

$-\tan\delta = -\dfrac{1}{8[(n-\frac{1}{4})\pi + \delta]}$

So expand to $O(\delta)$ on LHS

$$-\delta == -\frac{1}{8[(n-\frac{1}{4})\pi + \delta]}$$

Hence roots are $x = \left(n - \dfrac{1}{4}\right)\pi + = -\dfrac{1}{8[(n-\frac{1}{4})\pi + \delta]} + \cdots$

$\left(= o\left(\dfrac{1}{n}\right)\right), \quad n \to \infty$

(ii)

$n =$	1	2	3	4	5
exact =	2.40482..	5.52007..	8.65372..	11.79153..	14.93091..
approx. =	2.40925..	5.52052..	8.65385..	11.79158..	14.93094..
%error =	0.18%	0.008%	0.001%	0.0004%	0.0002%

$\left|\dfrac{\text{exact} - \text{approx}}{\text{exact}}\right| \times 100$

so it seems even $n = 1$ is a large parameter!!!