Question

Show that the small ε expansion of the roots of

$$x^2 - x - 2 + \frac{1}{2}\varepsilon(x^3 + 2x + 3) = 0$$

are

$$x = \begin{cases} -1 \\ 2 - \frac{5}{2}\varepsilon + O(\varepsilon^2) \\ -\frac{2}{\varepsilon} - 1 + O(\varepsilon) \end{cases}$$

Sketch the behaviour of the roots as $\varepsilon \to 0^+$.

Answer

$x^2 - x - 2 + \frac{\varepsilon}{2}(x^3 2 + 2x + 3) = 0$

Substitute $x = -1$ to check it's a root: $1 + 1 - 2 + \frac{\varepsilon}{2}(-1 - 2 + 3) = 0\surd$

When $\varepsilon = 0$, $x^2 - x - 2 = 0 \Rightarrow x = -1$ or $+2$, non degenerate.

Therefore can try $x = x_0 + \varepsilon x + O(\varepsilon^3 2)$ as ansatz.

Note ε multiplies highest power $\Rightarrow$ singular perturbation.

Therefore try $x - \frac{x_0}{\varepsilon} + x_1 + O(\varepsilon)$ as ansatz.

This completes our 3 roots:

<u>Regular</u>:

$[x_0 + \varepsilon x_1 + O(\varepsilon^2)]^2 - [x_0 + \varepsilon x_1 + O(\varepsilon^2)] - 2 + \frac{\varepsilon}{2}[x_0 + \varepsilon x_1 + O(\varepsilon^2)]^3 + \varepsilon(x_0 + \varepsilon x_1 + O(\varepsilon^2)) + \frac{3}{2}\varepsilon = 0$

$x_0^2 + 2\varepsilon x_0 x_1 - x_0 - \varepsilon x_1 - 2 + \frac{\varepsilon}{2}x_0^3 + \varepsilon x_0 + \frac{3}{2}\varepsilon + O(\varepsilon^2) = 0$

Therefore $(x_0^2 - x_0 - 2) + \varepsilon\left(2x_0 x_1 - x_1 + \frac{x_0^3}{2} + x_0 + \frac{3}{2}\right) + O(\varepsilon^2) = 0$

Balance at

$O(\varepsilon^0) : x_0^2 - x_0 - 2 \to x_0 = -1$ or $x_0 = +2$ as above

$O(\varepsilon^1) : 2x_0 x_1 - x_1 + \frac{x_0^3}{2} + x_0 + \frac{3}{2} = 0 \Rightarrow x_0 = -1 \Rightarrow x_1 = 0$ (as expected since $x_0 = -1$ is EXACT, or $x_0 = 2 \Rightarrow x_0 = \frac{5}{2}$

$$\Rightarrow \begin{cases} x = -1 \\ x = 2 - \frac{5}{2}\varepsilon + O(\varepsilon^2) \end{cases}$$

Singular root:

$$\left[\frac{x_0}{\varepsilon}+x_1+O(\varepsilon)\right]^2-\left[\frac{x_0}{\varepsilon}+x_1+O(\varepsilon)\right]-2$$
$$+\frac{1}{2}\varepsilon\left[\frac{x_0}{\varepsilon}+x_1+O(\varepsilon)\right]^3+\frac{\varepsilon}{2}\cdot 2\left[\frac{x_0}{\varepsilon}+x_1+O(\varepsilon)\right]+3\frac{\varepsilon}{2}=0$$
$$\frac{x_0^2}{\varepsilon^2}+2\frac{x_0x_1}{\varepsilon}-\frac{x_0}{\varepsilon}+\frac{x_0^3}{2^2}+\frac{3}{2}\frac{x_0^2x_1}{\varepsilon}+O(\varepsilon^0)=0$$

Balance at:

$O\left(\frac{1}{\varepsilon^2}\right)$: $x_0^2+\frac{x_0^3}{2}=0 \Rightarrow$ either $\left.\begin{matrix} x_0=0 \\ x_0=0 \end{matrix}\right\}$ (non singular roots) discard, or $x_0=-2$

$O\left(\frac{1}{\varepsilon}\right)$: $2\underline{x_0x_1}-x_0+\frac{3}{2}x_0^2x_1=0 \rightarrow x_0=-2 \Rightarrow x_0=-2 \Rightarrow x_0=-1$

Therefore root is $x=-\frac{2}{\varepsilon}-1+O(\varepsilon),\ \varepsilon\rightarrow 0^+$

From $-\infty$