

QUESTION

Show that if  $z_n = x_n + iy_n$  (for  $n = 1, 2, \dots$ ), then

$$\lim_{n \rightarrow \infty} z_n = z$$

if and only if

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y.$$

ANSWER

Suppose that  $\lim_{n \rightarrow \infty} x_n = x$ , and  $\lim_{n \rightarrow \infty} y_n = y$ . Then given  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $|x_n - x| < \epsilon/2$ ,  $|y_n - y| < \epsilon/2$ . If  $x + iy = z$ , and  $x_n + iy_n = z_n$  then  $|z_n - z| \leq |x_n - x| + |y_n - y| < \epsilon$ .  $\lim_{n \rightarrow \infty} z_n = z$ . For the converse we just use  $|\operatorname{Re}(z)| \leq |z|$  and  $|\operatorname{Im}(z)| \leq |z|$ .