Question

Suppose that X follows the exponential distribution, i.e. $f(x) = e^{-x},\ x > 0$. Obtain the mgf of this distribution and show that $E(X^k) = k!$ where k is a positive integer. Verify this result using direct integration.

Answer

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^\infty e^{tx}e^{-x}\,dx \\ &= \frac{1}{1-t}, \quad t < 1 \\ &= 1 + t + t^2 + t^3 + \ldots \text{ valid for } |t| < 1. \end{aligned}$$

However

$$\begin{aligned} E(X^k) &= \left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0} \\ &= \left. k! + \frac{(k+1)!}{1!}t + \frac{(k+1)!}{2!}t^2 + \ldots \right|_{t=0} \\ &= k! \end{aligned}$$

(See this by taking k=1,2,3, and so on)

Alternative:

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k e^{-x} f(x)\,dx \\ &= -x^k e^{-x}\Big|_0^\infty + \int_0^\infty kx^{k-1}e^{-x}\,dx \quad \text{[Integration by parts]} \\ &= k\int_0^\infty x^{k-1}e^{-x}\,dx \\ &\vdots \\ &= k!\int_0^\infty e^{-x}\,dx \\ &= k! \end{aligned}$$