Question

Suppose that a r.v. X, either discrete or continuous, has mean μ and variance σ^2. Define a function $h(c)$ by

$$h(c) = E[(X-c)^2], \quad c \in R^1.$$

By noting that $h(c)$ is a quadratic function is c, show that $h(c)$ attains its minimum value, σ^2, at $c = \mu$. By noting $E[(X-\mu)^2] \geq 0$, show that $E(X)^2 \leq E(X^2)$. If $E(X)^2 = E(X^2)$ what can be said about X?

Answer

Note that

$$\begin{aligned} h(c) &= E[(X-\mu+\mu-c)^2] \\ &= E[(X-\mu)^2 + 2(X-\mu)(\mu-c) + (\mu-c)^2] \\ &= \sigma^2 + (\mu-c)^2 \end{aligned}$$

from which it is clear that $h(c)$ attains its minimum σ^2 at $c = \mu$. $E(X^2) \geq \{E(X)\}^2$ follows obviously from
$var(X) = E(X^2) - \{E(X)\}^2 \geq 0$. If the quantity $var(X) = 0$ then X is constant and so X is non-random.