Question

If f and g are measurable functions such that $\int_E f = \int_E g$ for all measurable sets E then show that $f = g$ a.e.

(Hint. Suppose $f = g$ a.e. is false and consider the sequence of sets
$E_n = \{x | f(x) \geq g(x) + \frac{1}{n}\}$ or $F_n = \{x | g(x) \geq f(x) + \frac{1}{n}\}$)

Answer

Let $A_1 = \{x | f(x) > g(x)\}$ and $A_2 = \{x | f(x) < g(x)\}$
$A - 1 \cup A_2 = \{x | f(x) \neq g(x)\}$
Suppose $m(A_1) > 0, \;\; A_1 = \bigcup_{n=1}^{\infty} E_n$
Therefore there exists $n, \;\; mE_n > 0$
$E_n = \{x | f(x) - g(x) \geq \dfrac{1}{n}\}$ $\;\; f - g$ is measurable therefore E_n is measurable.
$\displaystyle\int_{E_n} f - \int_{E_n} g = \int_{E_n} f - g \geq \frac{1}{n} mE_n > 0$ contradiction.
Similarly if $m(A_2) > 0$ we get a contradiction.