QUESTION

(a) Consider the M/M/1/$n-2$ queue where $n \geq 2$. Let p_i be the steady state probability that there are i customers in the system. It is known that

$$p_{i+1} = \frac{\lambda}{\mu} p_i \quad \text{for } i = 0, 1, 2, \ldots, n-2.$$

Here, λ^{-1} is the mean inter-arrival time, and μ^{-1} is the mean service time.

(i) Show that

$$p_i = \frac{(1-\rho)\rho^i}{1-\rho^n}, \quad i = 0, 1, 2, \ldots, n-1.$$

Here, $\rho = \lambda/\mu$.

(ii) Find the expected number of customers in the system in terms of ρ and n.

(iii) Let L and W be the expected number of customers and the expected waiting time of customers in the system, respectively. Write down an expression which demonstrates Little's queueing formula.

(iv) Supposing that $\lambda/\mu < 1$, show that the expected waiting time of a customer in the system is $(\mu - \lambda)^{-1}$ as $n \to \infty$.

A repairman is to be hired to repair machines which break down at random at an average rate of 3 per day. Non-productive time on any machine costs the firm £100 per day. The firm can hire a slow cheap man charging £50 per day who can repair 4 machines per day on average. Alternatively, the firm can hire a fast expensive man charging £100 per day who can repair 6 machines per day on average. In either case, service times are exponentially distributed. Calculate the expected operation cost in both cases, and determine which repairman is more economical to hire.

ANSWER

(a) **(i)** From $p_{i+1} = \left(\frac{\lambda}{\mu}\right) p_i$ we have $p_i = \rho^i p_0$. Since $\sum_{i=0}^{n-1} p_i = 1$, we have

$$p_0 = \frac{1-\rho}{1-\rho^n}$$

(ii) The expected number of customers in the system is given by

$$
\begin{aligned}
L &= \sum_{i=0}^{n-1} i p_i \\
&= \sum_{i=1}^{n-1} i p_0 \rho^i \\
&= \frac{\rho - n\rho^n - (n-1)\rho^{n+1}}{(1-\rho)(1-\rho^n)}
\end{aligned}
$$

(iii) $L = \lambda W$ where L is the expected number of customers in the system and W is the expected waiting time for the customers.

(iv)

$$L = \lim_{n \to \infty} \frac{\rho - n\rho^n - (n-1)\rho^{n+1}}{(1-\rho)(1-\rho^n)} = \frac{\rho}{1-\rho}.$$

Therefore as $n \to \infty$ we have

$$W = \frac{L}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{(\mu - \lambda)}.$$

(b) One may regard the problem as an $M/M/1/\infty$ queuing problem with the customers being the broken machines.

In the case of slower repairman, $\lambda = 3$ and $\mu = 4$, therefore $\rho = 0.75$ and the expected number of broken machines in the system will be $\frac{\rho}{1-\rho}$. Therefore we have $\frac{\rho}{1-\rho} = 3$. Thus the operating cost will be

$$3 \times 100 + 50 = 350.$$

In the case of the faster repairman, $\lambda = 3$ and $\mu = 6$, therefore $\rho = 0.5$ and the expected number of broken machines in the system will be $\frac{\rho}{1-\rho} = 1$. Thus the operating cost will be

$$1 \times 100 + 100 = 200.$$

Hence it is more economical to employ the faster repairman.