QUESTION

Show that

$A = \begin{bmatrix} 3 & -2 & 1 \\ 3 & -1 & 1 \\ 4 & -5 & 4 \end{bmatrix}$

has a triple eigenvalue but only a one dimensional eigenspace (the line $x = \alpha, y = 2\alpha, z = 3\alpha$) so A cannot be diagonalised. Calculate $(A-2I)^2$ and $(A-2I)^3$.

Find a vector $\mathbf{u}$ satisfying

$(A-2I)^3\mathbf{u} = \mathbf{0}$ but $(A-2I)^2\mathbf{u} \neq \mathbf{0}$.

Calculate

$\mathbf{v} = (A-2I)\mathbf{u}$

and

$\mathbf{w} = (A-2I)\mathbf{v}$.

Show that $\mathbf{w}$ is an eigenvector of A.

Now form the matrix M the columns of which are the vectors $\mathbf{w},\mathbf{v},\mathbf{u}$ respectively and calculate $M^{-1}AM$

ANSWER

$A - 2I = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -3 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ $(A-2I)^2 = \begin{bmatrix} -1 & -1 & 1 \\ -2 & -2 & 2 \\ -3 & -3 & 3 \end{bmatrix}$ $(A-2I)^3 = \mathbf{0}$

Choosing $\mathbf{u}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then $\mathbf{v}=\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$. With this choice of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ then

$$M^{-1}AM = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 3 & -1 & 1 \\ 4 & -5 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & 0 \\ -3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$