Question

Prove, using the definition, that the function $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ given by $f(z) = \frac{2}{z^3}$ for $z \in \mathbf{C} - \{0\}$, $f(0) = \infty$, and $f(\infty) = 0$, is continuous.

Determine whether or not this map f is a homeomorphism of $\overline{\mathbf{C}}$.

Answer

Note that $f(z)$ is the composition of $J(z) = \dfrac{1}{z}$ $(J(0) = \infty, J(\infty) = 0)$ and the polynomial $g(z) = \dfrac{1}{2}z^3$ $(g(\infty) = \infty)$.
J is continuous by proposition 1.9 (in the book). g is continuous by exercise 1.14 (in the book), and composition of continuous functions is continuous (from class). So f is continuous.
f is <u>not</u> a homeomorphism, as it is not injective:
$f(z) = -1$ has 3 solutions, namely $-\sqrt[3]{2}$, $-\sqrt[3]{2}\omega$, and $-\sqrt[3]{2}\omega^2$ where $\omega = \exp\left(\dfrac{2\pi \mathrm{i}}{3}\right)$.