Question

Determine the hyperbolic line in the upper half plane that passes through the points $3+i$ and $-2+4i$.

Answer

Set $p = 3+i$ and $q = -2+4i$. Since Re(p)≠Re(q),the hyperbolic line through p and q lies in a euclidean circle of some center a and some radius r.

- calculate details of the (euclidean) line segment joining $p - q$:

  slope is $m = \dfrac{\mathrm{Im(p)} - \mathrm{Im(q)}}{\mathrm{Re(p)} - \mathrm{Re(q)}} = -\dfrac{3}{5}$

  midpoint is $\dfrac{1}{2}(p+q) = \dfrac{1}{2} + \dfrac{5}{2}i$

- calculate equation of perpendicular bisector:

  slope is $-\dfrac{1}{m} = \dfrac{5}{3}$ through $\dfrac{1}{2} + \dfrac{5}{2}i$ and so its equation is

  $y - \dfrac{5}{2} = \dfrac{5}{3}\left(x - \dfrac{1}{2}\right)$

- intersection with x-axis occurs at $(a, 0)$:

  $-\dfrac{5}{2} = \dfrac{5}{3}\left(a - \dfrac{1}{2}\right)$. So $-\dfrac{3}{2} = a - \dfrac{1}{2}$ and $\underline{a = -1}$.

- radius is $|a - p| = |q - a|$:

  $|a - p| = |-1 - 3 - i| = |-4 - i| = \sqrt{17}$

  $|a - q| = |-1 + 2 - 4i| = |1 - 4i| = \sqrt{17}$

  (as a check).

  So, this hyperbolic line is contained in the euclidean circle with center $a = -1$ and radius $r = \sqrt{17}$.