QUESTION
The following vectors form a basis for a three-dimensional subspace of $\mathbf{R}^4$:
(1,2,1,2), (0,3,4,5), (1,9,9,7).
Use the Gram-Schmidt process to find an orthonormal basis for the subspace.

ANSWER
Call the given vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and the desired vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, then

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{u}_1, \\
\mathbf{w}_2 &= \mathbf{u}_2 - \frac{(\mathbf{u}_2.\mathbf{w}_1)\mathbf{w}_1}{|\mathbf{w}_1|^2}, \\
\mathbf{w}_3 &= \mathbf{u}_3 - \frac{(\mathbf{u}_3\mathbf{w}_2)\mathbf{w}_2}{|\mathbf{w}_2|^2} - \frac{(\mathbf{u}_3.\mathbf{w}_1)\mathbf{w}_1}{|\mathbf{w}_1|^2},
\end{aligned}$$

Here

$$\begin{aligned}
\mathbf{w}_1 &= (1,2,1,2), \\
\mathbf{w}_2 &= (0,3,4,5) - \frac{\{(0,3,4,5).(1,2,1,2)\}(1,2,1,2)}{10} \\
&= (0,3,4,5) - 2(1,2,1,2) \\
&= (-2,-1,2,1), \\
\mathbf{w}_3 &= (1,9,9,7) - \frac{\{(1,9,9,7).(-2,-1,2,1)\}(-2,-1,2,1)}{10} \\
&- \frac{\{(1,9,9,7).(1,2,1,2)\}(1,2,1,2)}{10} \\
&= (1,9,9,7) - \frac{14}{10}(-2,-1,2,1) - \frac{42}{10}(1,2,1,2) \\
&= \frac{1}{5}(-2,10,10,-14).
\end{aligned}$$

Orthogonal basis: (1,2,1,2), (-2,-1,2,1), (-1,5,5,-7).
Orthonormal basis: $\frac{(1,2,1,2)}{\sqrt{10}}$, $\frac{(-2,-1,2,1)}{\sqrt{10}}$, $\sqrt{(-1,5,5,-7)}10$