QUESTION
Check that the vectors
$(1,-1,2,-1),(-2,2,3,2),(1,2,0,-1),(1,0,0,1)$
form an orthogonal basis for $\mathbf{R}^4$. Express each of the following vectors as a linear combination of these basis elements.

(a) $(1,1,1,1)$

(b) $(\sqrt{2},-3\sqrt{2},5\sqrt{2},-\sqrt{2})$

(c) $(-\frac{1}{3},\frac{2}{3},-\frac{1}{3},\frac{4}{3})$

ANSWER It is straightforward to check that $\mathbf{u.v}$=0 for each of the six distinct pairs. Four mutually orthogonal vectors in $\mathbf{R}^4$ must form a basis.
Now if $\mathbf{v}=\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\ldots+\lambda_n\mathbf{v}_n$ where $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$ is an orthogonal basis, then

$$\lambda_r=\frac{\mathbf{v}.\mathbf{v}_r}{\mathbf{v}_r.\mathbf{v}_r}=\frac{(\mathbf{v}.\mathbf{v}_r)}{|\mathbf{v}_r|^2},$$

The coefficients only are given below.

(a) $\frac{(1,1,1,1).(1,-1,2,-1)}{7}=\frac{1}{7},$
$\frac{(1,1,1,1).(-2,2,3,2)}{21}=\frac{5}{21},$
$\frac{(1,1,1,1).(1,2,0,-1)}{6}=\frac{1}{3},$
$\frac{(1,1,1,1).(1,0,0,1)}{2}=1$

(b) $\frac{(\sqrt{2},-3\sqrt{2},5\sqrt{2},-\sqrt{2}).(1,-1,2,-1)}{7}=\frac{15\sqrt{2}}{7},$
$\frac{(\sqrt{2},-3\sqrt{2},5\sqrt{2},-\sqrt{2}).(-2.2.3.2)}{21}=\frac{5\sqrt{2}}{21},$
$\frac{(\sqrt{2},-3\sqrt{2},5\sqrt{2},-\sqrt{2}).(1,2,0,-1)}{6}=\frac{-4\sqrt{2}}{6}=\frac{-2\sqrt{2}}{3},$
$\frac{(\sqrt{2},-3\sqrt{2},5\sqrt{2},-\sqrt{2}).(1,0,0,1)}{2}=\frac{1}{2},$

(c) $\dfrac{(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3}).(1, -1, 2, -1)}{7} = -\dfrac{3}{7}$,

$\dfrac{(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3}).(-2, 2, 3, 2)}{21} = \dfrac{11}{63}$,

$\dfrac{(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3}).(1, 2, 0, -1)}{6} = -\dfrac{1}{18}$,

$\dfrac{(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3}).(1, 0, 0, 1)}{2} = \dfrac{1}{2}$,