QUESTION The inner product is sometimes written $<\mathbf{u},\mathbf{v}>$ rather than $\mathbf{u}.\mathbf{v}$. In this notation the basic properties become

(a)	$<\mathbf{u},\mathbf{v}>=<\mathbf{v},\mathbf{u}>$	symmetry,
(b)	$<\mathbf{u}+\mathbf{v},\mathbf{w}>=<\mathbf{u},\mathbf{w}>+<\mathbf{v},\mathbf{w}>$	additivity,
(c)	$<\lambda\mathbf{u},\mathbf{v}>=\lambda<\mathbf{u},\mathbf{v}>$	homogeneity,
(d)	$<\mathbf{v},\mathbf{v}>\geq 0$ with $<\mathbf{v},\mathbf{v}>=0 \Leftrightarrow \mathbf{v}=\mathbf{0}$	positivity.

More generally if V is a vector space then a function $<>: V\times V\rightarrow \mathbf{R}$ which associates a real number with each pair of ordered vectors is called an inner product if the above four properties hold; V itself is called an inner product space.

The concepts of length of a vector, distance between vectors, angle between vectors, orthogonal bases etc. can be defined for such spaces and the Gram-Schmidt process still works.

Example The vector space P_2 of polynomials of degree less than or equal to two can be turned into an inner product space by defining

$$<p,q>=\int_{-1}^{1}p(x)q(x)\,dx.$$

One basis for P_2 is the set $\{1,x,x^2\}$ and since

$$|p|=\sqrt{<p,p>}=\sqrt{\{\int_{-1}^{1}(p(x))^2\,dx\}},$$

the length of the "vector" 1 is

$$\sqrt{\{\int_{-1}^{1}1^2\,dx\}}=\sqrt{\{[x]_{-1}^{1}\}}=\sqrt{2}$$

and the length of the "vector" x is

$$\sqrt{\left\{\int_{-1}^{1}x^2\,dx\right\}}=\sqrt{\left\{\left[\frac{x^3}{3}\right]_{-1}^{1}\right\}}=\sqrt{\frac{2}{3}}.$$

Furthermore since

$$<1,x>=\int_{-1}^{1}(1\times x)\,dx=[\frac{x^2}{2}]_{-1}^{1}=0$$

the functions 1 and x are orthogonal with respect to the inner product and the functions $\frac{1}{\sqrt{2}}$ and $\sqrt{\frac{3}{2}}x$ are orthonormal. Since x^2 is not orthogonal to 1, however, the basis $\{1,x,x^2\}$ is not an orthogonal basis.

Exercise Apply the Gram-Schmidt process to the basis $\{1, x, x^2\}$ to turn it into an orthogonal basis and the normalise the new basis. (The resulting polynomials are the first three normalised Legendre polynomials.)

ANSWER Since 1 and x are orthogonal one can take

$$\begin{aligned} \mathbf{w}_0 &= 1, \\ \mathbf{w}_1 &= x. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{w}_2 &= x^2 - \frac{[\int_{-1}^{1}(x^2 \times x)\,dx]x}{\frac{2}{3}} - \frac{[\int_{-1}^{1}(x^2 \times 1)\,dx]1}{2} \\ &= x^2 - \frac{[\frac{x^4}{4}]_{-1}^{1}x}{\frac{2}{3}} - \frac{[\frac{x^3}{3}]_{-1}^{1}}{2} \\ &= x^2 - 0 - \frac{1}{3} \\ &= \frac{(3x^2-1)}{3} \end{aligned}$$

Now

$$\begin{aligned} \mathbf{w}_2.\mathbf{w}_2 &= \int_{-1}^{1} \frac{(3x^2-1)^2}{9}\,dx \\ &= \int_{-1}^{1} \frac{(9x64 - 6x^2 + 1)}{9}\,dx \\ &= \frac{[\frac{9x^5}{5} - 2x^3 + x]_{-1}^{1}}{9} \\ &= \frac{2}{9}[\frac{9}{5} - 2 + 1] \\ &= \frac{8}{45}. \\ |\mathbf{w}_2| &= \frac{2\sqrt{2}}{3\sqrt{5}} \end{aligned}$$

The orthonormal basis giving the first few normalised Legendre polynomials is:

$$\hat{\mathbf{w}}_0 = \frac{1}{\sqrt{2}},$$

$$\begin{aligned}
\hat{\mathbf{w}}_1 &= \sqrt{\frac{3}{2}}x \\
\hat{\mathbf{w}}_2 &= \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)
\end{aligned}$$