Question
Consider the system of equations

$$\begin{aligned} 2x+3y+6z &= 1 \\ 6x+2y-3z &= 1 \\ 3x-6y+2z &= 2 \end{aligned}$$

Show by matrix inversion that the solution set is $x=\dfrac{2}{7},\ y=-\dfrac{1}{7},\ z=\dfrac{1}{7}$
What is the geometrical significance of this?
Note: No marks will be given unless a full working of the calculation of the inverse matrix is shown.

Answer

$$\begin{aligned} 2x+3y+6z &= 1 \\ 6x+2y-3z &= 1 \\ 3x-6y+2z &= 2 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\mathbf{A}\cdot \quad \mathbf{X} \quad = \quad \mathbf{K}$
Require $\mathbf{A}^{-1}$, since $\mathbf{X}=\mathbf{A}^{-1}\mathbf{K}$
Step (iii)

$$\begin{aligned} \triangle &= detA \\ &= \begin{vmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 2 & -3 \\ -6 & 2 \end{vmatrix} - 3\begin{vmatrix} 6 & -3 \\ 3 & 2 \end{vmatrix} + 6\begin{vmatrix} 6 & 2 \\ 3 & -6 \end{vmatrix} \\ &= 2\times(4-18)-3\times(12+9) \\ &= -3\times 42 \\ &= -28-63-252 \\ &= -343 \\ &\neq \underline{0} \end{aligned}$$

so there exists an inverse

Step (i) Cofactors of matrix are given by

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{pmatrix}$$

cofactor of $A_{11} = + \begin{vmatrix} 2 & -3 \\ -6 & 2 \end{vmatrix} = -14$ Following + - sign pattern

cofactor of $A_{12} = - \begin{vmatrix} 6 & -3 \\ 3 & 2 \end{vmatrix} = -21$

cofactor of $A_{13} = + \begin{vmatrix} 6 & 2 \\ 3 & -6 \end{vmatrix} = -42$

cofactor of $A_{21} = - \begin{vmatrix} 3 & 6 \\ -6 & 2 \end{vmatrix} = -42$

cofactor of $A_{22} = + \begin{vmatrix} 2 & 6 \\ 3 & 2 \end{vmatrix} = -14$

cofactor of $A_{23} = - \begin{vmatrix} 2 & 3 \\ 3 & -6 \end{vmatrix} = +21$

cofactor of $A_{31} = + \begin{vmatrix} 3 & 6 \\ 2 & -3 \end{vmatrix} = -21$

cofactor of $A_{32} = - \begin{vmatrix} 2 & 6 \\ 6 & -3 \end{vmatrix} = +42$

cofactor of $A_{33} = + \begin{vmatrix} 2 & 3 \\ 6 & 2 \end{vmatrix} = -14$

Matrix of cofactors is thus:

$$\begin{pmatrix} -14 & -21 & -42 \\ -42 & -14 & +21 \\ -21 & +42 & -14 \end{pmatrix}$$

Step (ii)

Transpose this to get $adjA$

$$adj\ A = \begin{pmatrix} -14 & -42 & -21 \\ -21 & -14 & 42 \\ -42 & 21 & -14 \end{pmatrix}$$

Step(iii)

$det\ A = -343$

Step (iv)

$$
\begin{aligned}
A^{-1} &= \frac{adj\ A}{det\ A} = \frac{1}{-343}\begin{pmatrix} -14 & -42 & -21 \\ -21 & -14 & 42 \\ -42 & 21 & -14 \end{pmatrix} \\
&= \frac{1}{49}\begin{pmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{pmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{X} &= -\frac{1}{49}\begin{pmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\
&= \frac{1}{49} \times \begin{pmatrix} 2+6+6 \\ 3+2-12 \\ 6-3+4 \end{pmatrix} \\
&= \frac{1}{49} \times \begin{pmatrix} 14 \\ -7 \\ 7 \end{pmatrix} \\
\mathbf{X} &= \underline{\begin{pmatrix} \frac{2}{7} \\ -\frac{1}{7} \\ \frac{1}{7} \end{pmatrix}}
\end{aligned}
$$

This the point of intersection of three planes in 3-D.