Question

Two geometrical objects are defined by the cartesian equations

$$2x + 7y + 5z = 3$$

$$\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-2}{-3}$$

One is a line, and the other is a plane. State which is which.

Convert the cartesian equations to vector equations and hence find the intersection between the objects.

Find the angle between the line and a normal to the plane.

Answer

$2x + 7y + 5z = 3$ is a plane

$\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-2}{-3}$ is a line

Plane can be rewritten as

$$(2, 7, 6) \cdot (x, y, z) = 3$$

for general $\mathbf{r} = (x, y, z)$

i.e.,

$$\underline{\mathbf{r} \cdot (2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) = 3}$$

For the line:

If, $\frac{x+3}{2} = \lambda$, $\frac{y-5}{-1} = \lambda$ and $\frac{z-2}{-3} = \lambda$ simultaneously

Therefore $\begin{cases} x &= 2\lambda - 3 \\ y &= -\lambda + 5 \\ z &= -3\lambda + 2 \end{cases}$

Therefore if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

$\mathbf{r} = (2\lambda - 3)\mathbf{i} + (5 - \lambda)\mathbf{j} + (2 - 3\lambda)\mathbf{k}$

Therefore $\underline{\mathbf{r} = -3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})}$

Intersection is when
$[(-3\mathbf{i}+5\mathbf{j}+2\mathbf{k})+\lambda(2\mathbf{i}-\mathbf{j}-3\mathbf{k})]\cdot(2\mathbf{BA}+7\mathbf{BA}+5\mathbf{BA})=3$
$\Rightarrow (2\lambda-3)\times 2+(5-\lambda)\times 7+(2-3\lambda)\times 5=3$
$\Rightarrow 4\lambda-6+35-7\lambda+10-15\lambda=3$
$\Rightarrow -18\lambda+39=3$
$\Rightarrow -18\lambda=-36$
$\Rightarrow \underline{\lambda=2}$
Therefore intersection at
$\mathbf{r}=(2\times 2-3)\mathbf{i}+(5-2)\mathbf{j}+(2-3\times 2)\mathbf{k}$

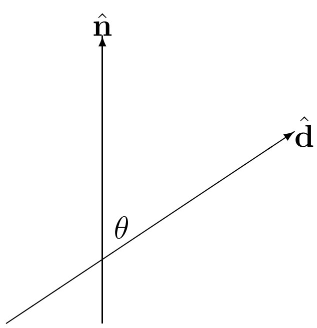
$$\underline{\mathbf{r}=\mathbf{i}+3\mathbf{j}-4\mathbf{k}}$$

Normal to plane is given by a vector

$$2\mathbf{i}+7\mathbf{j}+5\mathbf{k} \quad \text{or} \quad \frac{2\mathbf{i}+7\mathbf{j}+5\mathbf{k}}{\sqrt{4+49+25}}=\hat{\mathbf{n}}$$

Direction of line is given by vector

$$2\mathbf{i}-\mathbf{j}-3\mathbf{k} \quad \text{or} \quad \frac{2\mathbf{i}-\mathbf{j}-3\mathbf{k}}{\sqrt{4+1+9}}=\hat{\mathbf{d}}$$



So by scalar product

$$\begin{aligned}\mathbf{n}\cdot\mathbf{d} &= |\mathbf{n}||\mathbf{d}|\cos\theta\\ \text{or } \hat{\mathbf{n}}\cdot\hat{\mathbf{n}} &= \cos\theta\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\cos\theta &= \frac{(2,7,5)\cdot(2,-1,-3)}{\sqrt{4+49+25}\sqrt{4+1+9}} \\
&= \frac{-18}{\sqrt{78}\sqrt{14}} \\
&= -0.544705
\end{aligned}$$

$\Rightarrow \underline{\theta = 123.004^\circ}$ (or $180 - 123.004 = 56.9955^\circ$)