Question

Let the vertices of a triangle ABC have the following position vectors relative to some origin O:

$$\begin{aligned}\mathbf{OA} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k},\\ \mathbf{OB} &= \mathbf{i} + \mathbf{j} + \mathbf{k},\\ \mathbf{OC} &= \mathbf{j} + 2\mathbf{k}.\end{aligned}$$

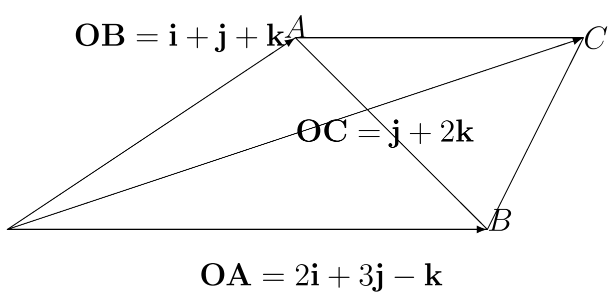
(i) Show these vectors and the triangle on a rough sketch.

(ii) Find the angle between **AB** and **AC**. Repeat the calculation for **BA** and **BC**. Hence deduce the three angles within the triangle.

(iii) Calculate the area of the triangle ABC using an appropriate vector product.

Answer

(i)



(ii)

$$\begin{aligned}\mathbf{AB} = \mathbf{AO} + \mathbf{OB} &= \mathbf{OB} - \mathbf{OA}\\ &= \mathbf{i} + \mathbf{j} + \mathbf{k} - 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\\ &= -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{AC} = \mathbf{AO} + \mathbf{OC} &= \mathbf{OC} - \mathbf{OA}\\ &= \mathbf{j} + 2\mathbf{k} - 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}\\ &= -2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\end{aligned}$$

$$\mathbf{AB} \cdot \mathbf{AC} = |\mathbf{AB}||\mathbf{AC}|\cos(\theta)$$

$|\mathbf{AB}| = \sqrt{1+4+4} = 3$

$|\mathbf{AC}| = \sqrt{4+4+9} = \sqrt{17}$

$\Rightarrow$

$$\begin{aligned} \cos(\theta) &= \frac{(-1,-2,2)\cdot(-2,-2,3)}{3\sqrt{17}} \\ &= \frac{2+4+6}{3\sqrt{17}} = \frac{4}{\sqrt{17}} \end{aligned}$$

$\theta = \arccos(0.970143) = 14.04°$

$\mathbf{BA} = -\mathbf{AB} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

$$\begin{aligned} \mathbf{BC} &= \mathbf{BO} + \mathbf{OC} = -\mathbf{OB} + \mathbf{OC} \\ &= -\mathbf{i} - \mathbf{j} - \mathbf{k} + \mathbf{j} + 2\mathbf{k} \\ &= -\mathbf{i} + \mathbf{k} \end{aligned}$$

$\mathbf{BA} \cdot \mathbf{BC} = |\mathbf{BA}||\mathbf{BC}|\cos\theta$

$|\mathbf{BA}| = \sqrt{1+4+4} = 3$

$|\mathbf{BC}| = \sqrt{1+1} = \sqrt{2}$

Therefore

$$\begin{aligned} \cos\theta &= \frac{(1,2,-2)\cdot(-1,0,1)}{3\sqrt{2}} \\ &= \frac{-1-2}{3\sqrt{2}} \\ &= -\frac{1}{\sqrt{2}} \\ \theta &= \underline{135°} \end{aligned}$$

Other angle $= \angle ACB = 180 - 135 - 14.04 = 30.86°$

(iii)

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\begin{aligned}
\text{Area of } \bigtriangledown \text{ABC} &= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 2 \\ -2 & -2 & 3 \end{vmatrix} \\
&= \frac{1}{2}|\mathbf{i}(-2 \times 3 - (-2) \times 2) \\
&\quad -\mathbf{j}((-1) \times 3 - (-2) \times 2) \\
&\quad +\mathbf{k}((-1) \times (-2) - (-2) \times (-2)| \\
&= \frac{1}{2}| - 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| \\
&= \frac{1}{2}\sqrt{4 + 1 + 4} \\
&= \underline{\frac{3}{2}}
\end{aligned}
$$