Question

(i) Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0.$$

(ii) Use the result of part (i) to find the solution of

$$\frac{d^2y}{dx^2} + 2\frac{Dy}{dx} - 3y = 10\sin x,$$

where $y = 0$ and $\dfrac{dy}{dx} = -5$ when $x = 0$.

Answer

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$

Trial solution $y = Ae^{mx}$

$\Rightarrow$ Auxiliary equation of CF

$$m^2 + 2m - 3 = 0$$

$\Rightarrow$

$$\begin{aligned} m &= \frac{-2 \pm \sqrt{4 - (4 \cdot 1 \cdot 3}}{2} \\ &= \frac{-2 \pm 4}{2} \\ &= 1 \text{ or } -3 \end{aligned}$$

Thus we have a solution

$$\underline{y = Ae^{x} + Be^{-3x}}$$

(ii) PI has form

$$y = A\sin x + B\cos x$$

Substitute into equation

$$\begin{aligned} y' &= A\cos x - B\sin x \\ y'' &= -A\sin x - B\cos x \end{aligned}$$

$\Rightarrow -A\sin x - B\cos x + 2A\cos x - 2B\sin x - 3A\sin x - 3B\cos x = 10\sin x$

$\Rightarrow \left.\begin{array}{rcl} -A - 2B - 3A &=& 10 \\ -B + 2A - 3B &=& 0 \end{array}\right\} \left\{\begin{array}{rcl} -4A - 2B &=& 10 \\ 2A - 4B &=& 0 \end{array}\right.$

$\Rightarrow A = 2B$

$$\begin{aligned} -4(2B) - 2B &= 10 \\ -8B - 2B &= 10 \\ B &= -1 \Rightarrow A = -2 \end{aligned}$$

Therefore PI is

$$\underline{y = -2\sin x - \cos x}$$

Whole solution is

$$\begin{aligned} y &= CF + PI \\ y &= Ae^{x} + Be^{-3x} - 2\sin x - \cos x \\ y' &= Ae^{x} - 3Be^{-3x} - 2\cos x + \sin x \end{aligned}$$

Use $y(0) = 0,\ y'(0) = -5$

$\left.\begin{array}{rcl} 0 &=& A + B - 2\times 0 - 1 \\ -5 &=& A - 3B - 2\times 1 + 0 \end{array}\right\} \left\{\begin{array}{rcl} 1 &=& A + B \\ -3 &=& A - 3B \end{array}\right.$

$\Rightarrow 4 = 4B \Rightarrow B = 1$ Therefore $A = 0$

Therefore

$$\underline{y = e^{-3x} - 2\sin x - \cos x}$$