Question
Given that

$$I_m = \int_0^{\frac{\pi}{2}} x^m \cos x \, dx \quad ,$$

integrate by parts twice to derive the reduction formula

$$I_m = \left(\frac{\pi}{2}\right)^m - m(m-1)I_{m-2}, \quad m \geq 2.$$

Show that $I_0 = 1$ and calculate I_4.

Answer
$I_m = \int_0^{\frac{\pi}{2}} x^m \cos x \, dx$

$u = x^m \qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = mx^{m-1} \quad v = +\sin x$

$I_m = [x^m \sin x]_0^{\frac{\pi}{2}} - m \int_0^{\frac{\pi}{2}} x^{m-1} \sin x \, dx$

$J_{m-1} = \int_0^{\frac{\pi}{2}} x^{m-1} \sin x \, dx$

$u = x^{m-1} \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (m-1)x^{m-2} \quad v = -\cos x$

$$\begin{aligned} J_m &= [-x^{m-1} \cos x]_0^{\frac{\pi}{2}} + (m-1) \int_0^{\frac{\pi}{2}} x^{m-2} \cos x \, dx \\ &= \left[\left(\frac{\pi}{2}\right)^{m-1} \cos \frac{\pi}{2} - 0^{m-1} \cos 0\right] + (m-1)I_{m-2} \end{aligned}$$

Therefore $I_m = \left[\left(\frac{\pi}{2}\right)^m \sin \frac{\pi}{2} - 0^m \sin 0\right] - m(m-1)I_{m-2}$

$\Rightarrow \underline{I_m = \left(\frac{\pi}{2}\right)^m - m(m-1)I_{m-2}}$

$$\begin{aligned} I_0 &= \int_0^{\frac{\pi}{2}} \cos x \, dx \\ &= [\sin x]_0^{\frac{\pi}{2}} \\ &= 1 - 0 \\ &= \underline{1} \end{aligned}$$

$I_4 = \left(\frac{\pi}{2}\right)^4 - 4 \times 3 I_2$
$I_2 = \left(\frac{\pi}{2}\right)^2 - 2 \times 1 I_0$
$I_0 = 1$
$\Rightarrow$

$$
\begin{aligned}
I_4 &= \left(\frac{\pi}{2}\right)^4 - 12\left[\left(\frac{\pi}{2}\right)^2 - 2\right] \\
I_4 &= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24 \\
&= \frac{\pi^4}{16} - 3\pi^2 + 24 \\
&= \underline{0.479255...}
\end{aligned}
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