Question

A sequence has its n^{th} term given by $u_n = \frac{3n-1}{4n-5}$. Write the 1^{st}, 5^{th}, 10^{th}, 100^{th}, 1000^{th}, $10,000^{\text{th}}$, and $100,000^{\text{th}}$ term of the sequence in decimal form. Make a **guess** as to the limit of this sequence as $n \to \infty$. Using the definition of limit, verify that the guess you've made is correct.

Answer

- $u_1 = \frac{3(1)-1}{4(1)-5} = \frac{2}{-1} = -2$;
- $u_5 = \frac{3(5)-1}{4(5)-5} = \frac{14}{15} \approx 0.9333$;
- $u_{10} = \frac{3(10)-1}{4(10)-5} = \frac{29}{35} \approx 0.8286$;
- $u_{100} = \frac{3(100)-1}{4(100)-5} = \frac{299}{395} \approx .7570$;
- $u_{1000} = \frac{3(1000)-1}{4(1000)-5} = \frac{2999}{3995} \approx 0.7507$;
- $u_{10000} = \frac{3(10000)-1}{4(10000)-5} = \frac{29999}{39995} \approx 0.7501$;
- $u_{100000} = \frac{3(100000)-1}{4(100000)-5} = \frac{299999}{399995} \approx 0.7500$;

So, it seems that a reasonable guess would be that $L = \lim_{n\to\infty} u_n$ exists and equals $0.75 = \frac{3}{4}$. To verify this, we use the definition: we need to show that for any choice of $\varepsilon > 0$, we can find M so that $|u_n - L| < \varepsilon$ for all $n > M$.

Calculating, we see that

$$|u_n - L| = \left|\frac{3n-1}{4n-5} - \frac{3}{4}\right| = \left|\frac{4(3n-1) - 3(4n-5)}{4(4n-5)}\right| = \left|\frac{11}{4(4n-5)}\right| = \frac{11}{4(4n-5)}.$$

(The last equality follows since $u_n - L$ is positive for $n > 1$.)

To find the value of M so that $|u_n - L| < \varepsilon$ for $n > M$, we start by solving for n: since $\frac{11}{4(4n-5)} < \varepsilon$, we have that $\frac{11}{4\varepsilon} < 4n - 5$, and so $\frac{11}{16\varepsilon} + \frac{5}{4} < n$. That is, for a specified value of ε, we can take $M = \frac{11}{16\varepsilon} + \frac{5}{4} = \frac{11+20\varepsilon}{16\varepsilon}$. Then, for any choice of $\varepsilon > 0$, we set $M = \frac{11+20\varepsilon}{16\varepsilon}$, and then if we take $n > M$, working backwards we have that $|u_n - L| < \varepsilon$.