Question

Prove that $\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$ for all $n \in \mathbf{N}$.

Now, consider the sequence given by $a_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln(n)$. Prove that $\{a_n\}$ is a decreasing sequence and that each a_n is positive. Conclude that the limit $\gamma = \lim_{n\to\infty} a_n$ exists. [This number γ is known as **Euler's constant**, and little is known about it. For instance, it is not known whether γ is rational or irrational.]

Answer

We start with the first part of the inequality, that $\frac{1}{n+1} < \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right)$. Set $f(x) = \ln\left(\frac{x+1}{x}\right) - \frac{1}{x+1}$ and $b_n = f(n)$. We want to show that $f(x) > 0$ for all $x \geq 1$. Calculating, we see that $f'(x) = -\frac{1}{x(x+1)^2} < 0$ for all $x > 0$. This implies that $f(x)$ is decreasing, and hence that $\{b_n\}$ is a monotonically decreasing sequence. Since $\lim_{n\to\infty} b_n = 0$, this yields that $b_n > 0$ for all n. (Because, if some $b_M < 0$, then since $\{b_n\}$ is a monotonically decreasing sequence, we would have that $b_{M+k} < b_M$ for all $k \geq 0$, and so $\lim_{n\to\infty} b_n$ would then be negative.) Since $b_n > 0$ for all n, we have that $\ln\left(\frac{n+1}{n}\right) > \frac{1}{n+1}$ for all n, as desired.

To handle the other part of the inequality, consider $c_n = \frac{1}{n} - \ln\left(\frac{n+1}{n}\right)$ and set $g(x) = \frac{1}{x} - \ln\left(\frac{x+1}{x}\right)$, so that $c_n = g(n)$. Since $g'(x) = -\frac{1}{x^2(x+1)}$ for all $x > 0$, we see that $\{c_n\}$ is monotonically decreasing. Again, since $\lim_{n\to\infty} c_n = 0$, we see that $c_n > 0$ for all n, and hence that $\frac{1}{n} > \ln\left(\frac{n+1}{n}\right)$ for all n, as desired.

It remains to show that $\{a_n\}$ is bounded below and monotonically decreasing. Since

$$a_{n+1} - a_n = \left(\sum_{k=1}^{n+1} \frac{1}{k}\right) - \ln(n+1) - \left(\sum_{k=1}^{n}\right) + \ln(n) = \frac{1}{n+1} - \ln(n+1) + \ln(n)$$
$$= \tfrac{1}{n+1} - \ln\left(\tfrac{n+1}{n}\right),$$

we see that $a_{n+1} - a_n < 0$ by the first part of the inequality. That is, $\{a_n\}$ is monotonically decreasing.

Since $\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right)$ for all n, we have that

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln(n) = 1 + \left(\sum_{k=1}^{n-1} \frac{1}{k+1}\right) - \ln(n) > 1 + \sum_{k=1}^{n-1} \ln\left(\frac{k+1}{k}\right) - \ln(n) = 1,$$

and so $\{a_n\}$ is bounded below.

Since $\{a_n\}$ is bounded above (since $a_n < a_1$ for all n, since it is a monotonically decreasing sequence) and bounded below, it is bounded. Since it is also monotonic, we have that $\{a_n\}$ converges.