Question The sequence scavenger hunt: for each of the following sequences $\{a_n\}$, do the following:

- Determine whether the sequence converges or diverges;
- if the sequence converges, determine its limit;
- if the sequence diverges, determine whether the sequence converges to ∞ or if the sequence converges to $-\infty$ or neither;

1. $a_n = (n+2)^{1/n}$;
2. $a_n = \frac{n^2+3n+2}{6n^3+5}$;
3. $a_n = (1+\frac{1}{n})^n$;
4. $a_n = \frac{\sin(n)}{3^n}$;
5. $a_n = \sqrt{2n+3} - \sqrt{n+1}$;
6. $a_n = \cos\left(\frac{n\pi}{4}\right)$;
7. $a_n = (1+\frac{1}{n})^{1/n}$;
8. $a_n = \ln(n)$;
9. $a_n = e^n$;
10. $a_n = \frac{\ln(n)}{\sqrt{n}}$;
11. $a_n = \left(1 - \frac{2}{n^2}\right)^n$;
12. $a_n = \frac{n^3}{10n^2+1}$;
13. $a_n = x^n$, where x is a constant with $|x| < 1$;
14. $a_n = \frac{c}{n^p}$, where $c \neq 0$ and $p > 0$ are constants;
15. $a_n = \frac{2n}{5n-3}$;
16. $a_n = \frac{1-n^2}{2+3n^2}$;
17. $a_n = \frac{n^3-n+7}{2n^3+n^2}$;
18. $a_n = 1 + \left(\frac{9}{10}\right)^n$;

19. $a_n = 2 - \left(-\frac{1}{2}\right)^n$;

20. $a_n = 1 + (-1)^n$;

21. $a_n = \frac{1+(-1)^n}{n}$;

22. $a_n = \frac{1+(-1)^n\sqrt{n}}{\left(\frac{3}{2}\right)^n}$;

23. $a_n = \frac{\sin^2(n)}{\sqrt{n}}$;

24. $a_n = \sqrt{\frac{2+\cos(n)}{n}}$;

25. $a_n = n\sin(\pi n)$;

26. $a_n = n\cos(\pi n)$;

27. $a_n = \pi^{-\sin(n)/n}$;

28. $a_n = 2^{\cos(\pi n)}$;

29. $a_n = \frac{\ln(2n)}{\ln(3n)}$;

30. $a_n = \frac{\ln^2(n)}{n}$;

31. $a_n = n\sin\left(\frac{1}{n}\right)$;

32. $a_n = \frac{\arctan(n)}{n}$;

33. $a_n = \frac{n^3}{e^{n/10}}$;

34. $a_n = \frac{2^n+1}{e^n}$;

35. $a_n = \frac{\sinh(n)}{\cosh(n)}$;

36. $a_n = (2n+5)^{1/n}$;

37. $a_n = \left(\frac{n-1}{n+1}\right)^n$;

38. $a_n = (0.001)^{-1/n}$;

39. $a_n = 2^{(n+1)/n}$;

40. $a_n = \left(\frac{2}{n}\right)^{3/n}$;

41. $a_n = (-1)^n (n^2+1)^{1/n}$;

42. $a_n = \frac{\left(\frac{2}{3}\right)^n}{\left(\frac{1}{2}\right)^n + \left(\frac{9}{10}\right)^n}$;

Answer

1. **converges:** whenever we are evaluating a limit in which the variable (in this case n) appears in both the base and the exponent, we follow the same basic procedure. First use the identity $x = \exp(\ln(x))$ to rewrite the term. Here,

$$a_n = (n+2)^{1/n} = \exp\left(\frac{\ln(n+2)}{n}\right).$$

Next, we check to see whether we are dealing with an indeterminate form. Since the limit $\lim_{n\to\infty} \frac{\ln(n+2)}{n}$ has the indeterminate form $\frac{\infty}{\infty}$, we may use l'Hopital's rule to evaluate

$$\lim_{n\to\infty} \frac{\ln(n+2)}{n} = \lim_{n\to\infty} \frac{1}{n+2} = 0.$$

Hence, $\{a_n\}$ converges to $e^0 = 1$.

2. **converges:** there is a standard way of evaluating the limit as $n \to \infty$ of a rational function in n (where a rational function is the quotient of two polynomials). First, locate the highest power of n that appears in either the numerator or the denominator, and then multiply both numerator and denominator by its reciprocal. Here, the highest power of n that appears is n^3, and so we calculate

$$a_n = \frac{n^2+3n+2}{6n^3+5} = \frac{n^2+3n+2}{6n^3+5} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \frac{\frac{1}{n} + \frac{3}{n^2} + \frac{2}{n^3}}{6 + \frac{5}{n^3}}.$$

We then use several properties of limits: that the limit of a quotient is the quotient of the limits, that the limit of a sum is the sum of the limits, and that $\lim_{n\to\infty} \frac{1}{n} = 0$. Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{3}{n^2} + \frac{2}{n^3}}{6 + \frac{5}{n^3}} = \frac{0}{6} = 0.$$

Hence, $\{a_n\}$ converges to 0.

3. **converges:** as above, we first rewrite the term using $x = \exp(\ln(x))$. Here,

$$a_n = \left(1 + \frac{1}{n}\right)^n = \exp\left(n \ln\left(1 + \frac{1}{n}\right)\right) = \exp\left(\frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}\right).$$

We then concentrate on the exponent and check to see whether we are dealing with an indeterminate form, which in this case we are, since both $\lim_{n\to\infty} \ln(1 + \frac{1}{n})$ and $\lim_{n\to\infty} \frac{1}{n}$ are equal to 0. Hence, we may apply l'Hopital's rule to evaluate

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1.$$

Hence, $\{a_n\}$ converges to $e^1 = e$.

4. **converges:** here we use the squeeze law. Since $-1 \leq \sin(n) \leq 1$ for all n, we have that $-\frac{1}{3^n} \leq \frac{\sin(n)}{3^n} \leq \frac{1}{3^n}$. Since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, we have that $\lim_{n\to\infty} -\frac{1}{3^n} = 0$ as well, and so $\{a_n\}$ converges to 0.

5. **diverges:** write

$$a_n = (\sqrt{2n+3} - \sqrt{n+1}) \cdot \frac{\sqrt{2n+3} + \sqrt{n+1}}{\sqrt{2n+3} + \sqrt{n+1}} = \frac{n+2}{\sqrt{2n+3} + \sqrt{n+1}}.$$

We now massage algebraically, in order to simplify:

$$\frac{n+2}{\sqrt{2n+3} + \sqrt{n+1}} \geq \frac{n+2}{2\sqrt{2n+3}} = \frac{n + \frac{3}{2} + \frac{1}{2}}{2\sqrt{2(n+\frac{3}{2})}} > \frac{n + \frac{3}{2}}{2\sqrt{2(n+\frac{3}{2})}} = \frac{1}{2\sqrt{2}}\sqrt{n + \frac{3}{2}}.$$

Since $\lim_{n\to\infty} \sqrt{n + \frac{3}{2}} = \infty$, we see by the comparison test that $\lim_{n\to\infty} a_n = \infty$, and so $\{a_n\}$ diverges.

6. **diverges:** for $n = 8k$, $a_{8k} = \cos\left(\frac{8k\pi}{4}\right) = 1$, while for $n = 8k+1$, $a_{8k+1} = \cos\left(\frac{(8k+1)\pi}{4}\right) = \frac{1}{\sqrt{2}}$. In particular, $|a_{8k} - a_{8k+1}| = \frac{1}{\sqrt{2}}$, and so the sequence fails the Cauchy criterion, and so diverges.

7. **converges:** write $a_n = \left(1 + \frac{1}{n}\right)^{1/n} = \exp\left(\frac{\ln\left(1+\frac{1}{n}\right)}{n}\right)$. Since $\lim_{n\to\infty} \ln(1+\frac{1}{n}) = 0$, we have that $\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n} = 0$ (by the squeeze law for instance, since $0 \leq \frac{\ln(1+\frac{1}{n})}{n} \leq \ln(1+\frac{1}{n})$ for $n \geq 1$). Hence, $\lim_{n\to\infty} \exp\left(\frac{\ln(1+\frac{1}{n})}{n}\right) = e^0 = 1$, and so $\{a_n\}$ converges to 1.

8. **diverges:** given $\varepsilon > 0$, we show that there exists M so that $a_n > \varepsilon$ for $n > M$. Since $a_n = \ln(n)$, this becomes $\ln(n) > \varepsilon$ for $n > M$. Exponentiating both sides of $\ln(n) > \varepsilon$, we get that $n > e^\varepsilon$ (and vice versa, that if $n > e^\varepsilon$, then $\ln(n) > \varepsilon$, since e^x is an increasing function), and so we can take $M = e^\varepsilon$.

9. **diverges:** very similar to the question just done. Given $\varepsilon > 0$, we show that there exists M so that $a_n > \varepsilon$ for $n > M$. Taking logs of both sides of $a_n = e^n > \varepsilon$, we get that $n > \ln(\varepsilon)$. So, we make take $M = \ln(\varepsilon)$.

10. **converges:** since $\lim_{n\to\infty} a_n$ has the indeterminate form $\frac{\infty}{\infty}$ (as both $\ln(n) \to \infty$ and $\sqrt{n} \to \infty$ as $n \to \infty$), we may apply l'Hopital's rule to see that

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} == \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$$

Hence, $\{a_n\}$ converges to 0.

11. **converges:** as always, we first rewrite each term as

$$a_n = \left(1 - \frac{2}{n^2}\right)^n = \exp\left(n \ln\left(1 - \frac{2}{n^2}\right)\right) = \exp\left(\frac{\ln\left(1 - \frac{2}{n^2}\right)}{\frac{1}{n}}\right).$$

As $n \to \infty$, the exponent reveals itself to have the indeterminate form $\frac{0}{0}$, and so we may evaluate using l'Hopital's rule:

$$\lim_{n\to\infty} \frac{\ln\left(1 - \frac{2}{n^2}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{2}{n^2}} \cdot \frac{4}{n^3}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{-\frac{4}{1-\frac{2}{n^2}}}{n} = 0.$$

Hence, $\{a_n\}$ converges to $e^0 = 1$.

12. **diverges:** we could use either l'Hopital's rule (since the limit has the indeterminate form $\frac{\infty}{\infty}$) or the standard trick for dealing with limits of rational functions (multiply numberator and denominator by the reciprocal of the highest power of n appearing anywhere in the term), but instead we massage algebraically:

$$a_n = \frac{n^3}{10n^2 + 1} > \frac{n^3}{10n^2 + 10n^2} = \frac{n}{20}.$$

Since $\{\frac{n}{20}\}$ diverges, the comparison test gives that $\{a_n\}$ diverges as well.

13. **converges:** it is a reasonable guess that $\{a_n = x^n\}$ converges to 0, which by definition means that given $\varepsilon > 0$, there exists M so that $|x^n - 0| = |x^n| < \varepsilon$ for $n > M$. For $x = 0$, this is true, since $\{x^n\}$ becomes the constant sequence $\{a_n = 0\}$. So, we can assume that $x \neq 0$. Taking ln of both sides of $|x^n| < \varepsilon$ and using that $|x^n| = |x|^n$, we get that $n \ln(|x|) < \ln(\varepsilon)$, and so $n > \frac{\ln(\varepsilon)}{\ln(|x|)}$. (The direction of the inequality changes since $|x| < 1$ and so $\ln(|x|) < 0$.) Hence, we may take $M = \frac{\ln(\varepsilon)}{\ln(|x|)}$. [Then, if $n > M = \frac{\ln(\varepsilon)}{\ln(|x|)}$, then $n \ln(|x|) < \ln(\varepsilon)$, and exponentiating we get that $|x|^n < \varepsilon$, as desired.)

14. **converges:** recall that $n^p \geq n$ and that $n \to \infty$ as $n \to \infty$, and so $n^p \to \infty$ as $n \to \infty$. Hence, $\{\frac{1}{n^p}\}$ converges to 0, and therefore $\{a_n = \frac{c}{n^p}\}$ converges to $c \cdot 0 = 0$.

15. **converges:** using the standard trick for rational functions, write

$$a_n = \frac{2n}{5n-3} = \frac{2n}{5n-3} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{2}{5 - \frac{3}{n}}.$$

As $n \to \infty$, $\frac{1}{n} \to 0$ and so $\{a_n\}$ converges to $\frac{2}{5}$.

16. **converges:** using the standard trick for rational functions, write

$$a_n = \frac{1-n^2}{2+3n^2} = \frac{1-n^2}{2+3n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{\frac{1}{n^2} - 1}{\frac{2}{n^2} + 3}.$$

As $n \to \infty$, $\frac{1}{n^2} \to 0$ and so $\{a_n\}$ converges to $-\frac{1}{3}$.

17. **converges:** using the standard trick for rational functions, write

$$a_n = \frac{n^3 - n + 7}{2n^3 + n^2} = \frac{n^3 - n + 7}{2n^3 + n^2} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \frac{1 - \frac{1}{n^2} + \frac{7}{n^3}}{2 + \frac{1}{n}}.$$

As $n \to \infty$, both $\frac{1}{n^2} \to 0$ and $\frac{1}{n} \to 0$, and so $\{a_n\}$ converges to $\frac{1}{2}$.

18. **converges:** by a previous part of this exercise, we know that $\{(\frac{9}{10})^n\}$ converges to 0, since $|\frac{9}{10}| < 1$, and so $\lim_{n\to\infty}(1+(\frac{9}{10})^n) = 1+\lim_{n\to\infty}(\frac{9}{10})^n = 1$.

19. **converges:** by a previous part of this exercise, we know that $\{(-\frac{1}{2})^n\}$ converges to 0, since $|-\frac{1}{2}| < 1$, and so $\lim_{n\to\infty}(2 - (-\frac{1}{2})^n) = 2 - \lim_{n\to\infty}(-\frac{1}{2})^n = 2$.

20. **diverges:** for n even, $a_n = 2$, while for n odd, $a_n = 0$. In particular, $|a_n - a_{n+1}| = 2$ for all n, and so the sequence fails the Cauchy criterion and hence diverges.

21. **converges:** note that $0 \le 1 + (-1)^n \le 2$ for all n, and so the squeeze law yields that since $\lim_{n\to\infty} \frac{2}{n} = 0$, we have that $\lim_{n\to\infty} a_n = 0$.

22. **converges:** we begin by noting that

$$0 \le \frac{1 + (-1)^n \sqrt{n}}{(\frac{3}{2})^n} \le \frac{2\sqrt{n}}{(\frac{3}{2})^n},$$

and so we'll concentrate on evaluating $\lim_{n\to\infty} \frac{2\sqrt{n}}{(\frac{3}{2})^n}$ and hope to be able to apply the squeeze law. Since $\lim_{n\to\infty} \frac{2\sqrt{n}}{(\frac{3}{2})^n}$ has the indeterminate form $\frac{\infty}{\infty}$, we may use l'Hopital's rule to evaluate

$$\lim_{n\to\infty} \frac{2\sqrt{n}}{(\frac{3}{2})^n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\ln(\frac{3}{2}) \exp(n \ln(\frac{3}{2}))} = \lim_{n\to\infty} \frac{1}{\ln(\frac{3}{2}) \sqrt{n} (\frac{3}{2})^n} = 0$$

(where we differentiate $(\frac{3}{2})^n$ by first writing it as $\exp(n \ln(\frac{3}{2}))$). Hence, we may use the squeeze law to see that $\{a_n\}$ converges to 0.

23. **converges:** since $0 \le \sin^2(n) \le 1$ for all n and since $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ (since $\sqrt{n} \to \infty$ as $n \to \infty$), the comparison test yields that $\frac{\sin^2(n)}{\sqrt{n}} \to 0$ as $n \to \infty$. That is, $\{a_n\}$ converges to 0.

24. **converges:** since $1 \le \sqrt{2 + \cos(n)} \le \sqrt{3}$ for all n and since $\frac{1}{n} \to 0$ as $n \to \infty$, the squeeze law yields that $\sqrt{\frac{2+\cos(n)}{n}} \to 0$ as $n \to \infty$. That is, $\{a_n\}$ converges to 0.

25. **converges:** since $\sin(\pi n) = 0$ for all integers n, this sequence is the constant sequence $a_n = n \cdot 0 = 0$ for all n. In particular, $\{a_n\}$ converges to 0.

26. **diverges:** since $\cos(\pi n) = (-1)^n$, this sequence can be rewritten as $a_n = (-1)^n n$. For $n \ge 1$, $|a_{n+1} - a_n| \ge 2$, and so the sequences fails the Cauchy criterion, and so diverges.

27. **converges:** since $-1 \le -\sin(n) \le 1$ for all n, we have that $-\frac{1}{n} \le -\frac{\sin(n)}{n} \le \frac{1}{n}$ for all n, and so $\{-\frac{\sin(n)}{n}\}$ converges to 0. Hence, $\{a_n\}$ converges to $\pi^0 = 1$.

28. **diverges:** for n even, $\cos(\pi n) = 1$ and for n odd, $\cos(\pi n) = -1$. In particular, $|a_{n+1} - a_n| = |2^1 - 2^{-1}| = \frac{3}{2}$ for all n, and so this sequences fails the Cauchy criterion, and hence $\{a_n\}$ diverges.

29. **converges:** we could use l'Hopital's rule, since $\lim_{n\to\infty} \frac{\ln(2n)}{\ln(3n)}$ has the indeterminate form $\frac{\infty}{\infty}$, but we proceed in a more low tech way. Use the laws of logarithms and a variant of the standard trick for rational functions, we rewrite

$$a_n = \frac{\ln(2n)}{\ln(3n)} = \frac{\ln(2) + \ln(n)}{\ln(3) + \ln(n)} = \frac{\ln(2) + \ln(n)}{\ln(3) + \ln(n)} \cdot \frac{\frac{1}{\ln(n)}}{\frac{1}{\ln(n)}} = \frac{1 + \frac{\ln(2)}{\ln(n)}}{1 + \frac{\ln(3)}{\ln(n)}}.$$

Since $\ln(n) \to \infty$ as $n \to \infty$, we have that both $\frac{\ln(2)}{\ln(n)}$ and $\frac{\ln(3)}{\ln(n)}$ go to 0 as $n \to \infty$, and so $\lim_{n\to\infty} a_n = 1$.

30. **converges:** since $\lim_{n\to\infty} \frac{\ln^2(n)}{n}$ has the indeterminate form $\frac{\infty}{\infty}$, we can use l'Hopital's rule:

$$\lim_{n\to\infty} \frac{\ln^2(n)}{n} = \lim_{n\to\infty} \frac{2\ln(n)\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2\ln(n)}{n}.$$

This limit still has the indeterminate form $\frac{\infty}{\infty}$, and we can apply l'Hopital's rule again to get

$$\lim_{n\to\infty} \frac{2\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1} = 0.$$

Hence, $\{a_n\}$ converges to 0.

31. **converges:** write

$$a_n = n \sin\left(\frac{1}{n}\right) = \frac{\sin(\frac{1}{n})}{\frac{1}{n}}.$$

Since $\lim_{n\to\infty} a_n$ has the indeterminate form $\frac{0}{0}$, we can apply l'Hopital's rule to get

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1.$$

Hence, $\{a_n\}$ converges to 1. (There is also a geometric argument for evaluating this limit, that can be found in Adams (p. 116, Theorem 7).)

32. **converges:** as $n \to \infty$, $\arctan(n) \to \frac{\pi}{2}$, and so $\lim_{n\to\infty} \frac{\arctan(n)}{n} = 0$. (This is an application of the squeeze law, since the numerator is bounded by 0 and π.)

33. **converges:** since $\lim_{n\to\infty} \frac{n^3}{e^{n/10}}$ has the indeterminate form $\frac{\infty}{\infty}$, we may use l'Hopital's rule:

$$\lim_{n\to\infty} \frac{n^3}{e^{n/10}} = \lim_{n\to\infty} \frac{3n^2}{\frac{1}{10}e^{n/10}}.$$

Since this latter limit still has the indeterminate form $\frac{\infty}{\infty}$, we use l'Hopital's rule again:

$$\lim_{n\to\infty} \frac{3n^2}{\frac{1}{10}e^{n/10}} = \lim_{n\to\infty} \frac{6n}{\frac{1}{100}e^{n/10}}.$$

And as we still have the indeterminate form $\frac{\infty}{\infty}$, we apply l'Hopital's rule yet again:

$$\lim_{n\to\infty} \frac{6n}{\frac{1}{100}e^{n/10}} = \lim_{n\to\infty} \frac{6}{\frac{1}{1000}e^{n/10}}.$$

The right hand limit evaluates to 0, and so $\{a_n\}$ converges to 0.

34. **converges:** write

$$a_n = \frac{2^n + 1}{e^n} = \frac{2^n}{e^n} + \frac{1}{e^n} = \frac{2^n}{e^n} + \frac{1^n}{e^n} = \left(\frac{2}{e}\right)^n + \left(\frac{1}{e}\right)^n.$$

Since both $\frac{2}{e} < 1$ and $\frac{1}{e} < 1$, we have that both $(\frac{2}{e})^n$ and $(\frac{1}{e})^n$ go to 0 as $n \to \infty$, and so their sum goes to 0 as $n \to \infty$. That is, $\{a_n\}$ converges to 0.

35. **converges:** again there are several possible approaches, including l'Hopital's rule, but again we take a low tech approach, and begin by expressing $\sinh(n)$ and $\cosh(n)$ in terms of e^n and e^{-n}, to get

$$a_n = \frac{\sinh(n)}{\cosh(n)} = \frac{e^n - e^{-n}}{e^n + e^{-n}} = \frac{e^n - e^{-n}}{e^n + e^{-n}} \cdot \frac{e^{-n}}{e^{-n}} = \frac{1 - e^{-2n}}{1 + e^{-2n}}.$$

Since $e^{-2n} = (\frac{1}{e^2})^n \to 0$ as $n \to \infty$, we see that $\lim_{n\to\infty} a_n = 1$. That is, $\{a_n\}$ converges to 1.

36. **converges:** as with all limits in which the variable appears in both the base and the exponent, we begin by rewriting using the identity $m = \exp(\ln(m))$ to get $a_n = (2n+5)^{1/n} = \exp\left(\frac{\ln(2n+5)}{n}\right)$. We may now use l'Hopital's rule to evaluate the limit of the exponent $\lim_{n\to\infty} \frac{\ln(2n+5)}{n}$ (as it has the indeterminate form $\frac{\infty}{\infty}$) to get

$$\lim_{n\to\infty} \frac{\ln(2n+5)}{n} = \lim_{n\to\infty} \frac{\frac{2}{2n+5}}{1} = 0.$$

Therefore, $\{a_n\}$ converges to $e^0 = 1$.

37. **converges:** as with all limits in which the variable appears in both the base and the exponent, we begin by rewriting using the identity $m = \exp(\ln(m))$ to get

$$a_n = \left(\frac{n-1}{n+1}\right)^n = \left(\frac{n+1-2}{n+1}\right)^n = \left(1 - \frac{2}{n+1}\right)^n = \exp\left(n \ln\left(1 - \frac{2}{n+1}\right)\right).$$

Since the exponent has the indeterminate form $0 \cdot \infty$ as $n \to \infty$, we rewrite it as

$$n \ln\left(1 - \frac{2}{n+1}\right) = \frac{\ln(1 - \frac{2}{n+1})}{\frac{1}{n}},$$

which as the indeterminate form $\frac{0}{0}$ as $n \to \infty$. We now apply l'Hopital's rule to evaluate

$$\lim_{n\to\infty} \frac{\ln\left(1 - \frac{2}{n+1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1-\frac{2}{n+1}} \cdot \frac{2}{(n+1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{-2n^2}{\left(1 - \frac{2}{n+1}\right) \cdot (n+1)^2} = -2.$$

Hence, $\{a_n\}$ converges to e^{-2}.

38. **converges:** since $-\frac{1}{n} \to 0$ as $n \to \infty$, we see that $\{a_n\}$ converges to $(0.001)^0 = 1$.

39. **converges:** as $n \to \infty$, $\frac{n+1}{n} = 1 + \frac{1}{n} \to 1$, and so $\{a_n\}$ converges to $2^1 = 2$.

40. **converges:** one way to evaluate this limit is to write $a_n = (\frac{2}{n})^{3/n} = \frac{2^{3/n}}{n^{3/n}}$ and to evaluate the limits of the numerator and denominator separately. To evaluate $\lim_{n\to\infty} 2^{3/n}$, all we need note is that $\lim_{n\to\infty} \frac{3}{n} = 0$, and so $\{2^{3/n}\}$ converges to $2^0 = 1$.

To evaluate $\lim_{n\to\infty} n^{3/n}$, we rewrite $n^{3/n}$ as $n^{3/n} = \exp(\ln(n)\frac{3}{n})$ and use l'Hopital's rule to evaluate $\lim_{n\to\infty} \frac{3\ln(n)}{n}$ (since it has the indeterminate form $\frac{\infty}{\infty}$). Using l'Hopital's rule, we get that

$$\lim_{n\to\infty} \frac{3\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{3}{n}}{1} = 0,$$

and so $\{n^{3/n}\}$ converges to $e^0 = 1$. Therefore,

$$\lim_{n\to\infty} \frac{2^{3/n}}{n^{3/n}} = \frac{\lim_{n\to\infty} 2^{3/n}}{\lim_{n\to\infty} n^{3/n}} = \frac{1}{1} = 1.$$

41. **diverges:** begin by ignoring the $(-1)^n$ and worrying about what happens to the rest of the term. Using the standard trick, massage to get $(n^2+1)^{1/n} = \exp(\frac{\ln(n^2+1)}{n})$. Since $\lim_{n\to\infty} \frac{\ln(n^2+1)}{n}$ has the indeterminate form $\frac{\infty}{\infty}$, we may use l'Hopital's rule to evaluate

$$\lim_{n\to\infty} \frac{\ln(n^2+1)}{n} = \lim_{n\to\infty} \frac{\frac{2n}{n^2+1}}{1} = 0,$$

and so

$$\lim_{n\to\infty} \exp\left(\frac{\ln(n^2+1)}{n}\right) = e^0 = 1.$$

So, putting the $(-1)^n$ back into the picture, we see that $\{a_n\}$ fails the Cauchy criterion: specifically, since $\{\frac{n^2+1}{n}\}$ converges to 1, for any $\varepsilon > 0$, there exists M so that $\left|\frac{n^2+1}{n} - 1\right| < \varepsilon$ for $n > M$. Choose $\varepsilon = \frac{1}{2}$, and note that for $n > M$, we get that $|a_n - a_{n+1}| > 1$, since one of a_n, a_{n+1} is within $\frac{1}{2}$ of 1 and the other is within $\frac{1}{2}$ of -1 (remember the alternating signs). So, $\{a_n\}$ diverges.

42. **converges:** we perform a bit of algebraic massage: note that

$$a_n = \frac{\left(\frac{2}{3}\right)^n}{\left(\frac{1}{2}\right)^n + \left(\frac{9}{10}\right)^n} < \frac{\left(\frac{2}{3}\right)^n}{\left(\frac{9}{10}\right)^n} = \left(\frac{20}{27}\right)^n.$$

Since $\left(\frac{20}{27}\right)^n \to 0$ as $n \to \infty$ (since $\frac{20}{27} < 1$), the comparison test yields that $\{a_n\}$ converges to 0 as well.