Question

The **Fibonacci sequence** $\{a_n \mid n \geq 0\}$ is formed by setting $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Consider the derived sequence $\{q_n = \frac{a_n}{a_{n-1}}\}$ of quotients of consecutive terms of the Fibonacci sequence. Show that if $\lim_{n\to\infty} q_n$ exists, then $\lim_{n\to\infty} q_n = \frac{1+\sqrt{5}}{2}$.

Answer

Suppose that $\{q_n\}$ converges and set $x = \lim_{n\to\infty} q_n$. Now, note that

$$q_n = \frac{a_n}{a_{n-1}} = \frac{a_{n-1} + a_{n-2}}{a_{n-1}} = 1 + \frac{a_{n-2}}{a_{n-1}} = 1 + \frac{1}{q_{n-1}}.$$

Hence,

$$x = \lim_{n\to\infty} q_n = \lim_{n\to\infty} \left(1 + \frac{1}{q_{n-1}}\right) = 1 + \frac{1}{\lim_{n\to\infty} q_{n-1}} = 1 + \frac{1}{x},$$

since $\lim_{n\to\infty} q_{n-1} = x$ as well. Therefore, $x = 1 + \frac{1}{x}$, and so (multiplying through by x and simplifying) x satisfies the quadratic equation $x^2 - x - 1 = 0$. By the quadratic formula, this yields that $x = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. However, since $q_n \geq 0$ for all n, it must be that $x \geq 0$ as well, and so $x = \frac{1}{2}\left(1 + \sqrt{5}\right)$.