Question
Prove that each of the following statements is true, using the definition of limit.

1. if $x_n \to -4$ as $n \to \infty$, then $\sqrt{|x_n|} \to 2$ as $n \to \infty$;
2. if $x_n \to -4$ as $n \to \infty$, then $x_n^2 \to 16$ as $n \to \infty$;
3. if $x_n \to -4$ as $n \to \infty$, then $\frac{x_n}{3} \to -\frac{4}{3}$ as $n \to \infty$;

Answer
In all three of these statements, we start with the same piece of information, namely that $\lim_{n\to\infty} x_n = -4$. That is, for each $\varepsilon > 0$, there exists M (which depends on ε) so that $|x_n - (-4)| = |x_n + 4| < \varepsilon$ for $n > M$.

1. we need to show that $\lim_{n\to\infty} \sqrt{|x_n|} = 2$, which is phrased mathematically as needing to show that for each $\mu > 0$, there exists P so that $|\sqrt{|x_n|} - 2| < \mu$ for $n > P$. We start by rewriting $|\sqrt{|x_n|} - 2|$, using the standard trick for handling differences of square roots, namely

   $$|\sqrt{|x_n|} - 2| = |\sqrt{|x_n|} - 2| \cdot \frac{|\sqrt{|x_n|} + 2|}{|\sqrt{|x_n|} + 2|} = \frac{|\,|x_n| - 4|}{|\sqrt{|x_n|} + 2|} \le \frac{|\,|x_n| - 4|}{2}.$$

   (The last inequality follows from the fact that $|\sqrt{|x_n|} + 2| \ge 2$ for all possible values of x_n.) Since for any $\mu > 0$, there exists M so that $|\,|x_n| - 4| < 2\mu$ (by using the definition of $\lim_{n\to\infty} |x_n| = 4$) for $n > M$, we have that

   $$|\sqrt{|x_n|} - 2| \le \frac{|\,|x_n| - 4|}{2} < \frac{2\mu}{2} = \mu$$

   for $n > M$, and so we are done.

2. we need to show that $\lim_{n\to\infty} x_n^2 = 16$, which is phrased mathematically as needing to show that for each $\mu > 0$, there exists P so that $|x_n^2 - 16| < \mu$ for $n > P$. We start by rewriting $|x_n^2 - 16|$, using that it is the difference of two squares:

   $$|x_n^2 - 16| = |(x_n - 4)(x_n + 4)| = |x_n - 4|\,|x_n + 4|.$$

   Now apply the definition of $\lim_{n\to\infty} x_n = -4$ with $\varepsilon = 1$, so that there exists M so that if $n > N$, then $|x_n - (-4)| < 1$. In particular, if $n > M$, then $-5 < x_n < -3$, and so $|x_n| < 5$, and so $|x_n - 4| \le |x_n| + 4 < 9$.

Since $x_n \to -4$ by assumption, we know that for any $\varepsilon > 0$, there is Q so that $|x_n - (-4)| = |x_n + 4| < \frac{1}{9}\varepsilon$ for $n > Q$. Hence, if $n > P = \max(M, Q)$, then

$$|x_n^2 - 16| = |x_n - 4|\,|x_n + 4| < 9\,\frac{1}{9}\varepsilon = \varepsilon,$$

as desired.

3. we need to show that $\lim_{n\to\infty} \frac{x_n}{3} = -\frac{4}{3}$, which is phrased mathematically as needing to show that for each $\mu > 0$, there exists P so that $|\frac{x_n}{3} - (-\frac{4}{3})| = |\frac{x_n}{3} + \frac{4}{3}| < \mu$ for $n > P$. Note that $|\frac{x_n}{3} - (-\frac{4}{3})| = |\frac{x_n}{3} + \frac{4}{3} = \frac{1}{3}|x_n + 4|$. We know from the definition of $\lim_{n\to\infty} x_n = -4$ given above that for any $\mu > 0$, there exists M so that $|x_n - (-4)| = |x_n + 4| < 3\mu$ for $n > M$. Hence, for $n > M$, we have that $\frac{1}{3}|x_n + 4| < \frac{1}{3}3\mu = \mu$ for $n > M$, and so we are done.