Question

Let $\{a_n\}$ be a sequence converging to a. Show that the following hold:

1. **square roots:** if $a > 0$, then $\{\sqrt{a_n}\}$ converges to $\sqrt{a}$;
2. $\{|a_n|\}$ converges to $|a|$;
3. if $a = \infty$, then $\{\frac{1}{a_n}\}$ converges to 0.
4. If $a \neq 0$, then $\{(-1)^n a_n\}$ diverges;
5. If $a = 0$, then $\{(-1)^n a_n\}$ converges to 0.

Answer

1. since $a > 0$, we can apply the definition of $\lim_{n\to\infty} a_n = a$ with $\varepsilon = \frac{1}{2}a$ to see that there exists P so that $a_n > 0$ for $n > P$ (since the interval of radius $\frac{1}{2}a$ centered at a contains only positive numbers), and so for $n > P$, $\sqrt{a_n}$ makes sense.

   We need to get our hands on $|\sqrt{a_n} - \sqrt{a}|$, which we do with our usual trick for handling differences of square roots:

   $$|\sqrt{a_n} - \sqrt{a}| = |\sqrt{a_n} - \sqrt{a}| \frac{|\sqrt{a_n} + \sqrt{a}|}{|\sqrt{a_n} + \sqrt{a}|} = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}}.$$

   (Here we're using that both $\sqrt{a_n} > 0$ and $\sqrt{a} > 0$ to say that $|\sqrt{a_n} + \sqrt{a}| = \sqrt{a_n} + \sqrt{a}$.) Since $\sqrt{a_n} + \sqrt{a} > \sqrt{a}$ for $n > P$, we have that

   $$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} < \frac{|a_n - a|}{\sqrt{a}}$$

   for $n > P$. Since $\{a_n\}$ converges to a, for every $\varepsilon > 0$, we can choose $M > P$ so that $|a_n - a| < \varepsilon\sqrt{a}$ for $n > M$. For this choice of M, we have that

   $$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} < \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon,$$

   and so $\{\sqrt{a_n}\}$ converges to $\sqrt{a}$.

2. this one, we break into three cases. If $a > 0$, then (applying the definition of $\lim_{n\to\infty} a_n = a$ with $\varepsilon = a$) there exists M_0 so that $a_n > 0$ for $n > M_0$. In this case, we have $|a_n| = a_n$ for $n > M_0$ and $|a| = a$, and so $||a_n| - |a|| = |a_n - a|$. Since there is M_1 so that $|a_n - a| < \varepsilon$ for

$n > M_1$, we have that $||a_n| - |a|| < \varepsilon$ for $n > M = \max(M_0, M_1)$, and so $\lim_{n\to\infty} |a_n| = |a|$.

If $a < 0$, then (applying the definition of $\lim_{n\to\infty} a_n = a$ with $\varepsilon = |a|$) there exists M_0 so that $a_n < 0$ for $n > M_0$. In this case, we have $|a_n| = -a_n$ for $n > M_0$ and $|a| = -a$, and so $||a_n| - |a|| = | - a_n + a| = |a_n - a|$. Since there is M_1 so that $|a_n - a| < \varepsilon$ for $n > M_1$, we have that $||a_n| - |a|| < \varepsilon$ for $n > M = \max(M_0, M_1)$, and so $\lim_{n\to\infty} |a_n| = |a|$.

If $a = 0$, then the definition of $\lim_{n\to\infty} a_n = a$ becomes: for every $\varepsilon > 0$, there exists M so that $|a_n - 0| = |a_n| < \varepsilon$ for $n > M$. Since $|\,|a_n|\,| = |a_n|$, we have that the definition of $\lim_{n\to\infty} |a_n| = 0$ is satisfied without any further work.

3. since $\lim_{n\to\infty} a_n = \infty$, for each $\varepsilon > 0$, there exists M so that $a_n > \varepsilon$ for $n > M$. Inverting both sides, we see that $\frac{1}{a_n} < \frac{1}{\varepsilon}$ for $n > M$. So, given $\mu > 0$, choose $\varepsilon > 0$ so that $\frac{1}{\varepsilon} < \mu$, which can be done by taking ε large enough. Then, there exists M so that $\left|\frac{1}{a_n} - 0\right| = \frac{1}{a_n} < \frac{1}{\varepsilon} < \mu$ for $n > M$, as desired.

4. if $a \neq 0$, consider the definition of $\lim_{n\to\infty} a_n = a$ with $\varepsilon = \frac{1}{2}|a|$: there exists M so that $|a_n - a| < \frac{1}{2}|a|$ for $n > M$. That is, a_n lies in the interval centered at a with radius $\frac{1}{2}|a|$, and so $|a_n| > \frac{1}{2}|a|$.

   Now consider the sequence $\{(-1)^n a_n\}$. For $n > M$ and n even, $(-1)^n a_n = a_n$ lies in the interval centered at a with radius $\frac{1}{2}|a|$. For $n > M$ and n odd, $(-1)^n a_n = -a_n$ lies in the interval centered at $-a$ with radius $\frac{1}{2}|a|$. In particular, we have, regardless of whether n is odd or even, that $|(-1)^n a_n - (-1)^{n+1} a_{n+1}| > |a|$ for $n > M$, since $(-1)^n a_n$ and $(-1)^{n+1} a_{n+1}$ lie on opposite sides of 0 and are both distance at least $\frac{1}{2}|a|$ from the origin. Hence, $\{(-1)^n a_n\}$ violates the Cauchy criterion (see Theorem below), and so diverges.

5. if $a = 0$, the definition of $\lim_{n\to\infty} a_n = 0$ becomes: for every $\varepsilon > 0$, there exists M so that $|a_n - 0| = |a_n| < \varepsilon$ for $n > M$. However, note that $|(-1)^n a_n - 0| = |a_n|$ as well, and so the definition of $\lim_{n\to\infty} (-1)^n a_n = 0$ is satisfied without any further work.

   Tests for convergence and divergence of sequences. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences.

   (a) **Comparison test:** If $a_n \leq b_n$ for all n and if $a_n \to \infty$ as $n \to \infty$, then $b_n \to \infty$ as $n \to \infty$;

(b) **Limit comparison test:** If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then $\{a_n\}$ converges if and only if $\{b_n\}$ converges.

(c) **l'Hopital's rule: l'Hopital's rule:** Suppose that f and g are differentiable on the union $I = (a-\varepsilon, a) \cup (a, a+\varepsilon)$ for some $\varepsilon > 0$, and that $g'(x)$ is non-zero on I. Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0.$$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided that the right hand limit either exists or is $\pm\infty$.

(d) **Squeeze rule:** If $a_n \leq b_n \leq c_n$ for all n and if $\{a_n\}$ and $\{c_n\}$ both converge with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $\{b_n\}$ converges with $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$.

(e) **Cauchy criterion:** if $\{a_n\}$ converges, then for every $\varepsilon > 0$, there exists M so that $|a_p - a_q| < \varepsilon$ for all p, $q > M$.