Question

Give five different examples of sequences that are bounded but not convergent.

Answer

1. $\{a_n = (-1)^n\}$, bounded above by 1 and bounded below by -1, hence bounded. This sequence fails the Cauchy criterion, since $|a_n - a_{n+1}| = 2$ for all n, and so diverges.

2. $\{\sin(n)\}$, bounded above by 1 and bounded below by -1, hence bounded. Though it seems fairly clear why this sequence diverges, the actual proof is a bit subtle, and we do not give it here.

3. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots\}$, bounded above by 1 and bounded below by 0, hence bounded. Arbitrarily far out in the sequence, there are consectutive terms taking the values 0 and 1, so the sequence fails the Cauchy criterion and hence diverges.

4. $\{a_n = \text{ the n}^{\text{th}} \text{ digit of } \pi\}$, bounded above by 9 and bounded below by 0, hence bounded. Does not converge, because the only way for a sequence of integers to converge is for it to be **eventually constant**, that is, constant past some index, which in this case would then imply that π is a repeating decimal, hence a rational number, which it isn't. (In fact, fixing an irrational number x and taking a_n to be the n^{th} digit of the decimal expansion of x gives a sequence that is bounded but not convergent, by the same argument.)

5. $\{a_n = \text{ the n}^{\text{th}} \text{ digit of the rational number } \frac{1}{7} = .\overline{142857}\}$, using the same argument as above (which works for rational numbers, as long as the length of the repeating section in the decimal expansion is longer than one digit).