Question

Let ℓ_0 and ℓ_1 be ultraparallel hyperbolic lines in $\mathbf{H}$. Label the endpoints at infinity of ℓ_0 as z_0 and z_1, and the endpoints at infinity of ℓ_1 as w_0 and w_1, so that they occur in the order z_0, w_0, w_1, and z_1 moving counter-clockwise around $\mathbf{R}$. Prove that

$$\tanh^2\left(\frac{1}{2}\mathrm{d}_{\mathbf{H}}(\ell_0, \ell_1)\right) = \frac{1}{1 - [z_0, w_0; w_1, z_1]}.$$

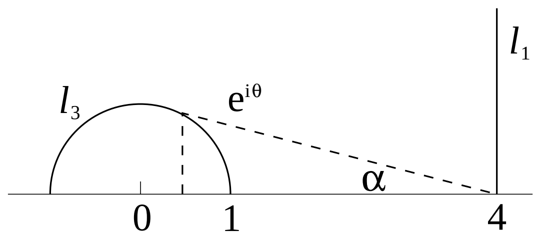
Answer

By the ordering of the parts around $\mathbf{R}$, there exists an element of Möb($\mathbf{H}$) taking z_0 to 0, z_1 to ∞, w_0 to 1, and w_1 to $x > 1$, so that

$$[z_0 w_0; w_1 z_1] = [0, 1; x, \infty] = \frac{x-1}{0-1} = 1 - x$$

and so $1 - [z_0 w_0; w_1 z_1] = x$.

$\underline{d_{\mathbf{H}}(\ell_0\ell_1)}$:



we used to determine the perpendicular bisector of $\ell_0\ell_1$:
By the euclidean pythagorean theorem:

$$\left(\frac{1+x}{2}\right)^2 = r^2 + \left(\frac{-1+x}{2}\right)^2$$
$$(1+x)^2 = 4r^2 + (x-1)^2$$
$$1 + 2x + x^2 = 4r^2 + x^2 - 2x + 1$$
$$4r^2 = 4x$$
$$r = \sqrt{x}$$
$$\cos(\alpha) = \frac{2r}{1+x} = \frac{2\sqrt{x}}{1+x}$$
$$\sin(\alpha) = \frac{x-1}{2} \cdot \frac{2}{x+1} = \frac{x-1}{x+1}$$

$$\begin{aligned} d_{\mathbf{H}}(\ell_0\ell_1) &= \int_{\alpha}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\, dt \\ &= -\ln|\csc(\alpha) - \cot(\alpha)| \\ &= \ln\left|\frac{\sin(\alpha)}{1-\cos(\alpha)}\right| = \ln\left|\frac{x-1}{1+x-2\sqrt{x}}\right| \end{aligned}$$

$$\begin{aligned} d_{\mathbf{H}}(\ell_0\ell_1) &= \ln\left(\frac{x-1}{(\sqrt{x}-1)^2}\right) \\ &= \ln\left(\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{(\sqrt{x}-1)^2}\right) \\ &= \ln\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) \end{aligned}$$

$$
\begin{aligned}
\tanh^2(a) &= \frac{\sinh^2(a)}{\cosh^2(a)} \\
&= \frac{(e^a - e^{-a})^2}{(e^a + e^{-a})^2} \\
&= \frac{e^{2a} + e^{-2a} - 2}{e^{2a} + e^{-2a} + 2}
\end{aligned}
$$

$$
\begin{aligned}
\tanh^2\left(\frac{1}{2} d_{\mathbf{H}}(\ell_0 \ell_1\right) &= \tanh^2\left(\frac{1}{2}\ln\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)\right) \\
&= \frac{\dfrac{\sqrt{x}+1}{\sqrt{x}-1} + \dfrac{\sqrt{x}-1}{\sqrt{x}+1} - 2}{\dfrac{\sqrt{x}+1}{\sqrt{x}-1} + \dfrac{\sqrt{x}-1}{\sqrt{x}+1} + 2} \\
&= \frac{(\sqrt{x}-1)^2 + (\sqrt{x}-1)^2 - 2(x-1)}{(\sqrt{x}+1)^2 + (\sqrt{x}-1)^2 + 2(x-1)} \\
&= \frac{4}{4x} = \frac{1}{x} = \frac{1}{1 - [z_0, w_0; w_1, z_1]}
\end{aligned}
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