Question
Given two sets X and Y in $\mathbf{H}$, define

$$\mathrm{d}_{\mathbf{H}}(X, Y) = \inf\{\mathrm{d}_{\mathbf{H}}(x, y) \mid x \in X, y \in Y\}.$$

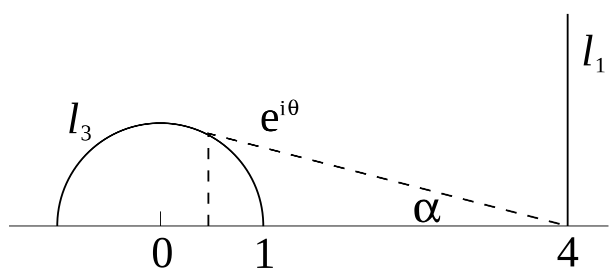
Now, let ℓ_1 be the hyperbolic line contained in the Euclidean line $\{\mathrm{Re}(z) = 4\}$, let ℓ_2 be the hyperbolic line contained in the Euclidean line $\{\mathrm{Re}(z) = -14\}$, and let ℓ_3 be the hyperbolic line contained in the Euclidean circle with Euclidean center 0 and Euclidean radius 1.

Calculate the three numbers $\mathrm{d}_{\mathbf{H}}(\ell_1, \ell_2)$, $\mathrm{d}_{\mathbf{H}}(\ell_2, \ell_3)$, and $\mathrm{d}_{\mathbf{H}}(\ell_1, \ell_3)$.

Use this to prove that $\mathrm{d}_{\mathbf{H}}(\cdot, \cdot)$ is *not* a metric on the set of subsets of $\mathbf{H}$.

Answer
$d_{\mathbf{H}}(\ell_1\ell_2) = 0$
Consider (for $\lambda > 0$) $4+\lambda \in \ell_1$ and $-14+\lambda \in \ell_2$. The horizontal euclidean line segment from $4+\lambda$ has hyperbolic length $\dfrac{18}{\lambda}$ and so $d_{\mathbf{H}}(4+\lambda, -14+\lambda) < \dfrac{18}{\lambda}$ (since the hyperbolic distance is the length of the hyperbolic line segment, which is less than the hyperbolic length of the euclidean line segment).
Hence $d_{\mathbf{H}}(\ell_1\ell_2) \le \inf\left\{ \dfrac{18}{\lambda} \middle| \lambda > 0 \right\} = 0$

For $d_{\mathbf{H}}(\ell_1\ell_3)$ and $d_{\mathbf{H}}(\ell_2\ell_3)$, these two use the same method:
$d_{\mathbf{H}}(\ell_1\ell_3)$ first draw the picture.



The distance from ℓ_1 to ℓ_3 is equal to the hyperbolic length of the common perpendicular.

Any hyperbolic line perpendicular to ℓ_1 is contained in a euclidean circle with center 4: it intersects ℓ_3 at $e^{i\theta}$ if (by the euclidean pythagorean theorem)

$$16 = 1 + |4 - e^{i\theta}|^2$$
$$16 = 1 + 16 - 4e^{i\theta} - 4e^{-i\theta} + 1$$
$$8\cos(\theta) = 2$$
$$\cos(\theta) = \frac{1}{4}$$
$$\sin(\theta) = \frac{\sqrt{15}}{4} \quad \theta \sim 1.3181...$$

Parametrize the common perpendicular by $f(t) = 4 + re^{it}$ where

$$\begin{aligned} r = |4 - e^{i\theta}| &= |4 - \cos(\theta) - i\sin(\theta)| \\ &= \sqrt{(4 - \frac{1}{4})^2 + \frac{15}{16}} \\ &= \frac{4\sqrt{15}}{4} = \sqrt{15} \end{aligned}$$

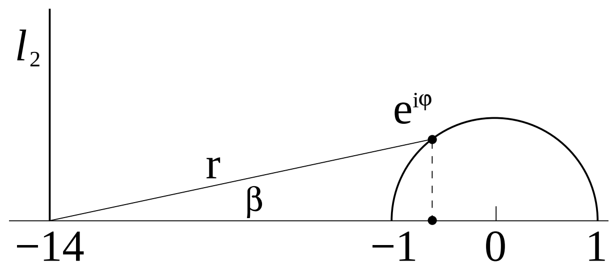
and where $\frac{\pi}{2} \le t \le \pi - \alpha$, where α is as in the picture.

$$\cos(\alpha) = \frac{1}{\sqrt{15}}(4 - \cos(\theta)) = \frac{\sqrt{15}}{4}$$
$$\sin(\alpha) = \frac{1}{\sqrt{15}}\sin(\theta) = \frac{1}{4}$$

So,

$$\begin{aligned} d_{\mathbf{H}}(\ell_1\ell_3) &= \int_{\frac{\pi}{2}}^{\pi-\alpha} \frac{1}{\sin(t)}\,dt \\ &= \ln|\csc(\pi - \alpha) - \cot(\pi - \alpha)| - \ln(\alpha)| \\ &= \ln|\csc(\alpha) + \cot(\alpha)| \\ &= \ln\left|\frac{1 + \cos(\alpha)}{\sin(\alpha)}\right| \\ &= \underline{\ln(4 + \sqrt{15})} \end{aligned}$$

$\underline{d_{\mathbf{H}}(\ell_2\ell_3)}$



ϕ determined by $|-14-e^{i\phi}|+1=14^2$

$$14^2+14e^{i\phi}+14e{-}i\phi+2=0$$

$$\begin{aligned} 28\cos(\phi) &= -2 \\ \cos(\phi) &= \frac{-1}{14} \end{aligned}$$

$$\sin(\phi)=\frac{\sqrt{195}}{14}$$

(so ϕ is a bit more than $\frac{\pi}{2}$, as indicated by the picture)
$r=|-14-e^{i\phi}=\sqrt{(14+\cos(\phi))^2+\sin^2(\phi)}=\sqrt{195}$

$$\begin{aligned} d_{\mathbf{H}}(\ell_2\ell_3) &= \int_{\beta}^{\frac{\pi}{2}} \frac{1}{\sin(t)}\,dt \\ &= \ln|\csc(\frac{\pi}{2})-\cot(\frac{\pi}{2})|-\ln|\csc\beta-\cot\beta| \\ &= -\ln\left|\frac{1-\cos(\beta)}{\sin(\beta)}\right| \\ &= \ln\left|\frac{\sin(\beta)}{1-\cos(\beta)}\right| \quad \cos(\beta)=\frac{\sqrt{195}}{14} \quad \sin(\beta)=\frac{1}{14} \\ &= \ln\left|\frac{1}{14-\sqrt{195}}\right| = \ln(14+\sqrt{195}) \end{aligned}$$

Note that since $d_{\mathbf{H}}(\ell_1\ell_2)=0$ but $\ell_1\neq\ell_2$, this cannot be a metric on the set of subsets of $\mathbf{H}$.