

Question

Prove that

$$\frac{1 + \cos \alpha + i \sin \alpha}{1 - \cos \alpha + i \sin \alpha} = \cot \frac{1}{2} \alpha \cdot \exp \left[i \left(\alpha - \frac{\pi}{2} \right) \right]$$

Answer

$$\begin{aligned} \frac{1 + e^{i\alpha}}{1 - e^{-i\alpha}} &= \frac{e^{i\frac{\alpha}{2}} \cdot e^{i\frac{\alpha}{2}} (e^{-i\frac{\alpha}{2}} + e^{i\frac{\alpha}{2}})}{e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}}} \\ &= \frac{e^{i\alpha} \cot \frac{\alpha}{2}}{i} \\ &= \cot \frac{\alpha}{2} e^{i(\alpha - \frac{\pi}{2})} \end{aligned}$$