Question

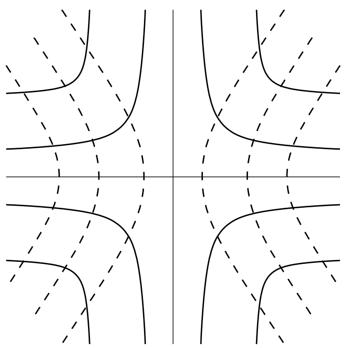
Consider the transformation $w = z^2$. Sketch the curves in the z-plane which map onto lines in the w-plane parallel to the real and imaginary axis.
Prove that every straight line in the w-plane is the image of a hyperbola (or pair of straight lines) in the z-plane.

Answer

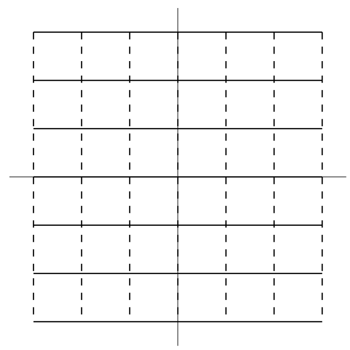
$z = x + iy \qquad w = \alpha + i\beta$

$$w = z^2 \text{ so } \begin{aligned} \alpha &= x^2 - y^2 \\ \beta &= 2xy \end{aligned}$$

$\beta = \text{constant} \Leftrightarrow 2xy = \text{constant}$
$\alpha = \text{constant} \Leftrightarrow x^2 - y^2 = \text{constant}$



$z = x + iy$



$w = \alpha + i\beta$

$\beta = m\alpha + c$ in w plane $\Leftrightarrow$
$2xy = m(x^2 - y^2) + c$
$mx^2 - my^2 - 2xy + c = 0$
"$b^2 - 4ac$" $= 4 + 4m^2 > 0$ Therefore a hyperbola or X