Question

Describe in geometrical terms the transformations defined by the following matrices. What effect do these transformations have on

(i) The square with vertices$(\pm 1, \pm 1)$,

(ii) the unit circle?

(a) $\begin{pmatrix} 4 & -6 \\ 6 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & 6 \\ -6 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

Answer

(a) $\begin{vmatrix} 4 & -6 \\ 6 & 4 \end{vmatrix} = 52$

$$\begin{pmatrix} 4 & -6 \\ 6 & 4 \end{pmatrix} = \sqrt{52} \begin{pmatrix} \frac{4}{\sqrt{52}} & \frac{-6}{\sqrt{52}} \\ \frac{6}{\sqrt{52}} & \frac{4}{\sqrt{52}} \end{pmatrix} = 52$$

So the matrix performs a magnification by a factor $\sqrt{52}$ and a rotation anticlockwise through $\cos^{-1} \frac{4}{\sqrt{52}}$

$$\begin{pmatrix} 4 & -6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 10 & -10 & 2 \\ 10 & 2 & -2 & -10 \end{pmatrix}$$

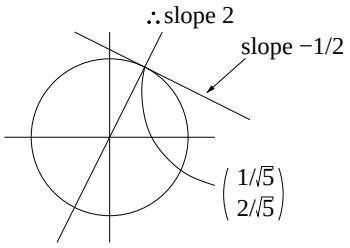
$x^2 + y^2 = 1 \rightarrow X^2 + Y^2 = 52$

(b) $\begin{pmatrix} 4 & 6 \\ -6 & 4 \end{pmatrix} = \sqrt{52}\begin{pmatrix} \frac{4}{\sqrt{52}} & \frac{6}{\sqrt{52}} \\ \frac{-6}{\sqrt{52}} & \frac{4}{\sqrt{52}} \end{pmatrix} = 52$

So the matrix performs a magnification by a factor $\sqrt{52}$ and a rotation clockwise through $\cos^{-1}\frac{4}{\sqrt{52}}$

(c) $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ Y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+4y \end{pmatrix}$ So $\mathbf{R}^2 \to Y = 2X$

The inverse image of $(k, 2k)$ is the line $x + 2y = k$. For the square the extremities of the image are $(\pm 3, \pm 6)$



So the extremities are
$\left(\pm\frac{5}{\sqrt{5}}, \pm\frac{10}{\sqrt{5}}\right) = \left(\pm\sqrt{5}, \pm 2\sqrt{5}\right)$

(d) Rotation through 45° anitclockwise

(e) $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 & -1 \\ 2 & 0 & 0 & -2 \end{pmatrix}$

Gives a magnification and shear.

$x = \frac{1}{2}X \qquad y = -\frac{1}{2}X + Y$

$x^2 + y^2 = 1 \to \frac{1}{2}X^2 - XY + Y^2 = 1$ - ellipse.