QUESTION

Solve the following differential equations subject to the given initial conditions.

(i) $(x+1)\frac{dy}{dx} = x+6$, where $y=3$ when $x=0$.

(ii) $x\frac{dy}{dx} + (3x+1)y = e^{-3x}$, where $y = e^{-3}$ when $x=1$.

(iii) $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$, where $y=1$ and $\frac{dy}{dx}=3$ when $x=0$.

(iv) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$, where $y=1$ and $\frac{dy}{dx}=1$ when $x=1$.

ANSWER

(i)

$$\begin{aligned} \frac{dy}{dx} = \frac{x+6}{x+1} = 1 + \frac{5}{x+1} \\ \Rightarrow \quad y = x + 5\ln|x+1| + c \; 3 = 5\ln 1 + c \text{ so } c = 3 \\ \Rightarrow \quad y = x + 5\ln|x+1| + 3 \end{aligned}$$

(ii) $\frac{dy}{dx} + \left(3 + \frac{1}{x}\right) y = \frac{1}{xe^{3x}}$

Integrating factor: $e^{\int 3+\frac{1}{x}\,dx} = e^{(3x+\ln|x|)} = |x|e^{3x}$. So for $x > 0$,

$$\frac{d(xe^{3x}y)}{dx} = \frac{1}{xe^{3x}}.xe^{3x} \Rightarrow xe^{3x}y$$

$$\int 1\,dx \Rightarrow y = \frac{x+C}{xe^{3x}} \;:\; e^{-3} = \frac{c+1}{e^3} \Rightarrow C = 0$$

so $y = \frac{1}{e^{3x}}$

(iii) Auxiliary equation: $2\lambda^2 - 5\lambda - 3 = 0$ factorises as $(2\lambda+1)(\lambda-3)$ so $\lambda = 3, -\frac{1}{2}$ with solutions $y = Ae^{3x} + Be^{-\frac{1}{2}x}$. $1 = A+B,\; 3 = 3A - \frac{1}{2}B$ from initial conditions so $3\frac{1}{2}B = 0 \Rightarrow B = 0$ and $A = 1$ giving $y = e^{3x}$.

(iv) Auxiliary equation: $\lambda^2 - 10\lambda + 25 = 0$ factorises as $\lambda - 5)^2$ so there is a unique solution $\lambda = 5$. $y = (A+Bx)e^{5x}$.

$$\frac{dy}{dx} = Be^{5x} + 5(A+Bx)e^{5x} = (5A+B+5Bx)e^{5x}$$

$1 = (A+B)e^5$ and $1 = (5A+6B)e^5$ from the initial conditions so $5 = Ae^5$ or $A = 5e^{-5}$ and so $Be^5 = 1 - 5e^{-5}$ i.e $B = -4e^{-5}$ and $y = (5-4x)e^{5(x-1)}$