QUESTION

(a) Sketch the plane region R defined by the inequalities $3x^2 + 2y^2 \leq 1$ and $x \geq 0$ and $y \geq 0$ and evaluate the following double integral:

$$\iint_R xy \, d(x, y).$$

(b) Sketch the region S defined by the inequalities $a^2 \leq x^2 + y^2 \leq b^2$ and $y \leq x$ and $y \geq 0$, where a and b are positive real numbers. Evaluate the integral $\iint_S \frac{y^2}{x^2} \, d(x, y)$. (HINT: You may use the fact that $\tan^2\theta = \sec^2\theta - 1$.)

ANSWER

(a) DIAGRAM

$$\begin{aligned}
\iint_R xy \, d(x, y) &= \int_{x=0}^{\frac{1}{\sqrt{3}}} \int_{y=0}^{\sqrt{\frac{1-3x^2}{2}}} xy \, dy dx \\
&= \int_0^{\frac{1}{\sqrt{3}}} \left[\frac{zy^2}{2} \right]_{y=0}^{y=\sqrt{\frac{1-3x^2}{2}}} dx \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{4 - 3x^3}{4} \, dx \\
&= \left[\frac{x^2}{8} - \frac{3x^4}{16} \right]_0^{\frac{1}{\sqrt{3}}} \\
&= \left(\frac{1}{24} - \frac{1}{48} \right) = \frac{1}{48}
\end{aligned}$$

(b) DIAGRAM

$$\begin{aligned}
\iint_S \frac{y^2}{x^2} \, d(x, y) &= \int_{\theta=0}^{\frac{\pi}{4}} \int_{\rho=a}^{b} \frac{\rho^2 \sin^2\theta}{\rho^2 \cos^2\theta} \rho \, d\rho d\theta \\
&= \int_{\theta=0}^{\frac{\pi}{4}} \left[\frac{\rho^2}{2} \tan^2\theta \right]_{\rho=a}^{b} d\theta \\
&= \frac{b^2 - a^2}{2} \int_{\theta=0}^{\frac{\pi}{4}} \sec^2 - 1 \, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b^2 - a^2}{2} \left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}} \\
&= \frac{b^2 - a^2}{2} \left(1 - \frac{\pi}{4}\right) \\
&= \frac{(4 - \pi)(b^2 - a^2)}{8}
\end{aligned}
$$