Question

Show that the 2-cycle that exists for $g_a(x) = ax(1-x)$ when $a > 3$ changes from attracting to repelling at $a = 1 + \sqrt{6}$.

Answer

If the period-2 orbit is $\{p, q\}$ then $g_a^{2\prime}(p) = g_a{}'(p)g_a{}'(q) = a^2(1-2p)(1-2q) = a^2(1-2(p+q)+4pq)$. Now $p,\ q$ (together with $0,\ 1-\frac{1}{a}$: fixed points of g_a) are the roots of $g_a^2(x) = x$, i.e. $a(ax(1-x))(1-ax(1-x)) = x$
i.e. $x(a^2(1-x)(1-ax+ax^2)-1) = 0$,
i.e. $x = 0$ or $a^3x^3 - 2a^3x^2 + a^2(1+a)x + (1-a^2) = 0$.
Sum of roots: $2 = p + q + (1 - \frac{1}{a})$, so $p + q = \frac{1}{a} + 1$.
Product of roots: $\frac{-1-a^2}{a^3} = pq\left(1-\frac{1}{a}\right)$, so $pq = \frac{1}{a}\left(\frac{1}{a}+1\right)$.
So from above we see

$$\begin{aligned} g_a^{2\prime}(p) &= a^2\left(1-2\left(\frac{1}{a}+1\right)+\frac{4}{a}\left(\frac{1}{a}+1\right)\right) \\ &= -a^2 - 2a + 4 + 4a \\ &= -a^2 + 2a + 4. \end{aligned}$$

Period-2 orbit $\{p, q\}$ becomes repelling when $g_a^{2\prime}(p) = -1$, i.e. $-a^2+2a+4 = -1$: $a = 1 \pm \sqrt{6}$ (only the + sign is relevant).