Question

Show that the following are exact and find the solution in each case.

1. $(xy^2 + y)dx + (x^2y + x)dy = 0$
2. $(e^x \sin y + 2x)dx + (e^x \cos y + 2y)dy = 0$
3. $e^{xy}(1 + xy)dx + x^2e^{xy}dy = 0 \qquad (*)$
4. $\left(2x + 1 - \frac{y^2}{x^2}\right) dx + \frac{2y}{x}dy = 0$

Answer

a) $p = xy^2 + y, \quad q = x^2y + x, \quad \frac{\partial P}{\partial y} = 2xy + 1, \quad \frac{\partial q}{\partial x} = 2xy + 1,$ exact.

$\frac{\partial F}{\partial x} = xy^2 + y \Rightarrow F = \frac{1}{2}x^2y^2 + xy + f(y) \Rightarrow \frac{\partial F}{\partial y} = x^2y^2 + x + \frac{df}{dy}$

$\frac{\partial F}{\partial y} = x^2y + x$ hence $\frac{df}{dy}(y) = 0, \quad f(y) = c.$

so the solution is $\frac{1}{2}x^2y^2 + xy = A.$

b) $p = e^x \sin y + 2x, \quad q = e^x \cos y + 2y,$

$\frac{\partial P}{\partial y} = e^x \cos y, \quad \frac{\partial Q}{\partial x} = e^x \cos y,$ exact.

$\frac{\partial F}{\partial x} = e^x \sin y + 2x \Rightarrow F = e^x \sin y + x^2 + f(y)$

$\Rightarrow \frac{\partial F}{\partial y} = e^x \cos y + \frac{df}{dy}$

$\frac{\partial F}{\partial y} = e^x \cos y + 2y \Rightarrow \frac{df}{dy} = 2y \Rightarrow f(y) = y^2 + c$

so the solution $e^x \sin y + x^2 + y^2 = A$

c) $p = e^{xy}(1 + xy), \quad q = x^2e^{xy}$

$\frac{\partial p}{\partial y} = xe^{xy}(1 + xy) + e^{xy}x, \quad \frac{\partial q}{\partial x} = 2xe^{xy} + x^2ye^{xy}$, hence it is exact.

$\frac{\partial F}{\partial x} = e^{xy}(1+xy), \quad \frac{\partial F}{\partial y} = x^2 e^{xy} \Rightarrow F = xe^{xy} + g(x)$

$\Rightarrow \frac{\partial F}{\partial x} = e^{xy} + xye^{xy} + \frac{dy}{dx}$ hence need $\frac{dy}{dx} = 0 \Rightarrow g(x) = c$

So the solution is $xe^{xy} = A \Rightarrow y = \frac{1}{x} \ln\left(\frac{A}{x}\right)$.

d) $p = 2x + 1 - \frac{y^2}{x^2}, \quad q = \frac{2y}{x}, \quad \frac{\partial p}{\partial y} = -\frac{2y}{x^2}, \quad \frac{\partial q}{\partial x} = -\frac{2y}{x^2}$, exact.

$\frac{\partial F}{\partial x} = 2x + 1 - \frac{y^2}{x^2}, \quad \frac{\partial F}{\partial y} = \frac{2y}{x} \Rightarrow F = \frac{y^2}{x} + g(x)$

$\Rightarrow \frac{\partial F}{\partial x} = -\frac{y^2}{x^2} + \frac{dy}{dx}$ hence need $\frac{dy}{dx} = 2x + 1 \Rightarrow g = x^2 + x + c$

so the solution is $\frac{y^2}{x} + x^2 + x = A$