Question

If $ad - bc \neq 0$, show that constants h and k can be chosen in such a way that the change of variables $X = x + h$, $Y = y + k$ reduces

$$\frac{dy}{dx} = \frac{ax + by + p}{cx + dy + q}$$

to a homogeneous equation.

Hence,or otherwise, solve the equation

$$\frac{dy}{dx} = \frac{x + y + 4}{x - y - 6}$$

Answer

The ODE becomes: $\dfrac{dY}{dX} = \dfrac{a(X-h) + b(Y-k) + p}{c(X-h) + d(Y-k) + q}$

$\dfrac{dY}{dX} = \dfrac{aX + bY + (p - ah - bk)}{cX + dY + (q - ch - dk)}$ for homogeneous we need

$\left.\begin{array}{l} p - ah - bk = 0 \\ q - ch - dk = 0 \end{array}\right\}$ which has a unique solution if $ad - bc \neq 0$

and the equation then becomes $\dfrac{dY}{dX} = \dfrac{a + b\frac{Y}{X}}{c + d\frac{Y}{X}}$

For example if $a = 1, \quad b = 1, \quad p = 4, \quad c = 1, \quad d = -1, \quad q = -6$

choose $h,\ k$ so that $\left.\begin{array}{l} h + k = 4 \\ h - k = -6 \end{array}\right\} \Rightarrow h = -1, \quad k = 5$

$X = x - 1, \quad Y = y + 5$ and the ODE becomes

$\dfrac{dY}{dX} = \dfrac{1 + \frac{Y}{X}}{1 - \frac{Y}{X}}$ put $Y = Xv \Rightarrow X\dfrac{dv}{dX} + v = \dfrac{1+v}{1-v}$

$X\dfrac{dv}{dX} = \dfrac{1 + v^2}{1 - v} \Rightarrow \displaystyle\int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{X}\,dX$

$\Rightarrow \displaystyle\int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \ln X + c$

$\Rightarrow \tan^{-1} v - \dfrac{1}{2}\ln(1 + v^2) = \ln X + c$

or $\tan^{-1}\dfrac{Y}{X} = \ln(X^2 + Y^2)^{\frac{1}{2}} + c$

$\Rightarrow \tan^{-1}\left(\dfrac{y+5}{x-1}\right) = \ln(x^2 + y^2 - 2x + 10y + 26)^{\frac{1}{2}} + c$