Question
Show that

$$(x^2+y^2-x)dx - ydy = 0$$

is not exact, but that it becomes exact if multiplied by $(x^2+y^2)^{-1}$. Hence obtain the solution of the differential equation $y\dfrac{dy}{dx} = x^2+y^2-x$. $\quad (*)$

Answer
$p = x^2+y^2-x, \quad q = -y \Rightarrow \dfrac{\partial p}{\partial y} = 2y, \quad \dfrac{\partial q}{\partial x} = 0 \Rightarrow$ Not exact.
Consider $(1 - \dfrac{x}{x^2+y^2}dx - \dfrac{y}{x^2+y^2}dy = 0, \Rightarrow p = 1 - \dfrac{x}{x^2+y^2},$
$q = -\dfrac{y}{x^2+y^2}, \quad \dfrac{\partial p}{\partial y} = \dfrac{2xy}{(x^2+y^2)^2}, \quad \dfrac{\partial q}{\partial x} = \dfrac{2xy}{(x^2+y^2)^2} \Rightarrow$ exact.
$\dfrac{\partial F}{\partial x} = 1 - \dfrac{x}{x^2+y^2} \Rightarrow F(x,y) = x - \dfrac{1}{2}\ln(x^2+y^2) + f(y)$
$\Rightarrow \dfrac{\partial F}{\partial y} = \dfrac{-y}{x^2+y^2} + \dfrac{df}{dy}$
$\dfrac{\partial F}{\partial y} = -\dfrac{y}{x^2+y^2}$ hence $\dfrac{df}{dy} = 0, \quad f(y) = c$
so the solution is $x - \dfrac{1}{2}\ln(x^2+y^2) = A$