Question

Obtain the general solution of the following differential equations:

1. $x\dfrac{dy}{dx} + y = 2e^{-x} \qquad (*)$

2. $(y - x^3) + (x + y^3)\dfrac{dy}{dx} = 0$

3. $x\dfrac{dy}{dx} = 3y + x^4$

4. $\dfrac{dy}{dx} = \dfrac{x^2 - xy + y^2}{xy}$

5. $\dfrac{dy}{dx} + y\cot x = 2x\ \mathrm{cosec}x \qquad (*)$

6. $(3x^2y^2 + \sin x)\dfrac{dy}{dx} + (2xy^3 + y\cos x) = 0$

Answer

a) $\dfrac{dy}{dx} + \dfrac{1}{x}y = 2\dfrac{1}{x}e^{-x} \Rightarrow I(x) = \exp(\int \dfrac{1}{x}dx) = \exp(\ln x) = x$

$\dfrac{d}{dx}(xy) = 2e^{-x} \Rightarrow xy = -2e^{-x} + c \Rightarrow y = \dfrac{c - 2e^{-x}}{x}$

b) $p = y - x^3, \quad q = x + y^3, \quad \dfrac{\partial p}{\partial y} = 1, \ \dfrac{\partial q}{\partial x} \Rightarrow$ exact.

$\dfrac{\partial F}{\partial x} = y - x^3 \Rightarrow F = xy - \dfrac{1}{4}x^4 + f(y) \Rightarrow \dfrac{\partial F}{\partial y} = x + \dfrac{df}{dy}$

$\dfrac{\partial F}{\partial y} = x + y^3$ hence need $\dfrac{\partial f}{\partial y} = y^3$ so $f = \dfrac{1}{4}y^4 + c.$

There is a solution if $F(x, y) = xy - \dfrac{1}{4}x^4 + \dfrac{1}{4}y^4 = A$

c) $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^3 \Rightarrow I(x) = \exp(\int -\dfrac{3}{x}dx) = \exp(-3\ln x) = x^{-3}$

$\dfrac{d}{dx}\left(\dfrac{1}{x^3}y\right) = 1 \Rightarrow \dfrac{1}{x^3}y = x + c \Rightarrow y = x^4 + cx^3$

d) $\dfrac{dy}{dx} = \dfrac{1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2}{\frac{y}{x}}$, put $y = xv \Rightarrow x\dfrac{dv}{dx} + v = \dfrac{1 - v + v^2}{v}$

$$x\frac{dv}{dx} = \frac{1 - v + v^2}{v} - v = \frac{1 - v}{v} \Rightarrow \int \frac{v}{1 - v} dv = \int \frac{1}{x} dx$$

$$\int -1 + \frac{1}{1 - v} dv = \ln|x| + c \Rightarrow -v - \ln|1 - v| = \ln|x| + c$$

$$e^{-v}\frac{1}{1 - v} = Ax \Rightarrow e^{-\frac{y}{x}}\frac{1}{x - y} = A$$

e) $\dfrac{dy}{dx} + \dfrac{\cos x}{\sin x} y = 2x\dfrac{1}{\sin x} \Rightarrow I(x) = exp\left(\int \dfrac{\cos x}{\sin x} dx\right)$

$$I(x) = \exp(\ln(\sin x)) = \sin x \Rightarrow \frac{d}{dx}((\sin x)y) = 2x$$

$$\Rightarrow (\sin x)y = x^2 + c \Rightarrow y = \frac{x^2 + c}{\sin x}$$

f) $p = 2xy^3 + y\cos x, \quad q = 3x^2y^2 + \sin x$

$$\frac{\partial P}{\partial y} = 6xy^2 + \cos x, \quad \frac{\partial q}{\partial x} = 6xy^2 + \cos x \Rightarrow \text{exact.}$$

$$\frac{\partial F}{\partial x} = 2xy^3 + y\cos x \Rightarrow F = x^2y^3 + y\sin x + f(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = 3x^2y^2 + \sin x + \frac{df}{dy}$$

$$\frac{\partial F}{\partial y} = 3x^2y^2 + \sin x \text{ hence need } \frac{df}{dy} = 0 \Rightarrow f = c$$

solution is $F(x, y) = x^3y^3 + y\sin x = A$