QUESTION

Given that the numbers 719, 853 and 971 are all prime, calculate $\left(\frac{719}{853}\right)$ and $\left(\frac{853}{971}\right)$.

ANSWER

$\left(\frac{719}{853}\right) = \left(\frac{853}{719}\right)$ by the quadratic reciprocity law, as $853 \equiv 1 \bmod 4$. Hence $\left(\frac{719}{853}\right) = \left(\frac{853}{719}\right) = \left(\frac{134}{719}\right) = \left(\frac{2}{719}\right)\left(\frac{67}{719}\right)$ (reducing mod 179 and factorising). Now $\left(\frac{2}{719}\right) = 1$ as $719 \equiv 7 \equiv -1 \bmod 8$ (th.7.3), while $\left(\frac{67}{719}\right) = -\left(\frac{719}{67}\right)$ by quadratic reciprocity, as 719 and 67 are both $\equiv 3 \bmod 4$. On reducing mod 67 we see that $\left(\frac{67}{719}\right) = -\left(\frac{719}{67}\right) = -\left(\frac{49}{67}\right) = -\left(\frac{7^2}{67}\right) = -1$ (as 7^2 is clearly a square!) Thus $\left(\frac{719}{853}\right) = 1. - 1 = -1$ and 719 is not a square mod 853.

[An alternative approach is to begin by replacing 719 by its least absolute residue -134 mod 853, and using $\left(\frac{719}{853}\right) = (-134853) = \left(\frac{-1}{853}\right)\left(\frac{134}{853}\right)$ etc. You might like to try this to see which method is quicker.]

$\left(\frac{853}{971}\right) = \left(\frac{971}{853}\right)$ by the quadratic reciprocity law, as $853 \equiv 1 \bmod 4$. Thus, on reducing mod 853, and factoring, we get $\left(\frac{853}{971}\right) = \left(\frac{971}{853}\right) = \left(\frac{118}{853}\right) = \left(\frac{2}{853}\right)\left(\frac{59}{853}\right)$. Now $\left(\frac{2}{853}\right) = -1$ as $853 \equiv 5 \equiv -3 \bmod 8$ (th.7.3). As 59 is prime, we may again assume quadratic reciprocity, and as $853 \equiv 1 \bmod 4$, we get $\left(\frac{59}{853}\right) = \left(\frac{853}{59}\right) = \left(\frac{27}{59}\right) = \left(\frac{3^2}{59}\right)\left(\frac{3}{59}\right) = \left(\frac{3}{59}\right)$ (by reducing mod 59, factoring and using $\left(\frac{a^2}{p}\right) = 1$.) Now 3 is prime, and as 59 and 3 are both congruent to 3 mod 4, quadratic reciprocity gives $\left(\frac{3}{59}\right) = -\left(\frac{59}{3}\right) = -\left(\frac{2}{3}\right) = -(-1) = 1$ (using th.7.3 again).

Thus $\left(\frac{59}{853}\right) = 1$ and so $\left(\frac{853}{971}\right) = (-1).1 = -1$, and 853 is not a square mod 971.