QUESTION

Explain why $\sum_{a=0}^{p-1} \left(\frac{a}{p}\right) = 0$ (where $\left(\frac{a}{p}\right)$ is the Legendre symbol.)

ANSWER

We know $\left(\frac{0}{p}\right) = 0$ by definition. We also know that of the $p-1$ non-zero residues mod p, exactly half of them are squares (viz. those which are even powers of a primitive root), and the rest are non-squares. Thus $\left(\frac{a}{p}\right) = 1$ for exactly $\frac{(p-1)}{2}$ values of a with $1 \leq a \leq p-1$, and $\left(\frac{a}{p}\right) = -1$ for the remaining $\frac{(p-1)}{2}$ values.

Hence $\sum_{a=0}^{p-1} \left(\frac{a}{p}\right)$ is a sum consisting of one zero, $\frac{(p-1)}{2}$ $+1$'s and $frac(p-1)2$ -1's. Thus it is 0.