QUESTION

Prove that $a^{49} - a$ is divisible by 70 for every integer a.

ANSWER

70=2.7.5. By the corollary to Fermat's little theorem (Corollary 4.3), $a^7 \equiv a \mod 7$, $a^5 \equiv a \mod 5$ and $a^2 \equiv a \mod 2$ for all a. Hence $a^{49} \equiv (a^7)^7 \equiv a^7 \equiv a \mod 7$, $a^{49} \equiv (a^5)^9 a^4 \equiv a^9.a^4 \equiv a^13 \equiv (a^5)^2 a^3 \equiv a^2 a^3 \equiv a^5 \equiv a \mod 5$, and $a^{49} \equiv (a^2)^2 4a \equiv a^2 4.a \equiv (a^2)^{12} a \equiv a^1 2.a \equiv (a^2)^6 a \equiv a^6 a \equiv (a^2)^3 a \equiv a^3.a \equiv a^4 \equiv (a^2)^2 \equiv a^2 \equiv a \mod 2$.

Thus $a^4 7 - a$ is divisible by 7,5 and 2, and hence (cor.1.7) by their product 70.