Question

Write down and solve the difference equations for the Gambler's ruin problem when the chances of winning and losing 1 unit are p and q, respectively, and the chance of a draw on a bet is r $(p+q+r=1)$.
(Hint: try, as a particular solution to the difference equation for the expected duration of the game, a multiple of the gambler's capital z).

Answer

$$\begin{aligned}
P(\text{ruin}) &= P(\text{ruin \& wins first bet}) + P(\text{ruin \& loses first bet}) \\
&\quad + P(\text{ruin \& draws on first bet}) \\
&= P(\text{ruin} \mid \text{wins first bet}) \cdot P(\text{wins first bet}) \\
&\quad + P(\text{ruin} \mid \text{loses first bet}) \cdot P(\text{loses first bet}) \\
&\quad + P(\text{ruin} \mid \text{draws first bet}) \cdot P(\text{draws first bet})
\end{aligned}$$

So

$$\begin{aligned}
q_z &= q_{z+1} \cdot p + q_{z-1} \cdot q + q_z \cdot r \\
q_z(1-r) &= pq_{z+1} + qq_{z-1} \\
q_z &= \frac{p}{p+q} q_{z+1} + \frac{q}{p+q} q_{z-1} \qquad \text{assuming } r \neq 1
\end{aligned}$$

with boundary conditions $q_0 = 1$ and $q_a = 0$
Substituting $q_z = \lambda^z$ gives

$$\lambda^z = \frac{p}{p+q}\lambda^{z+1} + \frac{q}{p+q}\lambda^{z-1}$$

$$\begin{aligned}
\text{i.e. } \frac{p}{p+q}\lambda^2 - \lambda + \frac{q}{p+q} &= 0 \\
(\lambda - 1)\left(\frac{p}{p+q}\lambda - \frac{q}{p+q}\right) &= 0
\end{aligned}$$

$$\text{So } \lambda = 1 \quad \text{or} \quad \lambda = \frac{q}{p}$$

IF $p \neq q$ then $q_z = A + B\left(\frac{q}{p}\right)^z$

The boundary conditions give $1 = A + B$ and $0 = A + B\left(\frac{q}{p}\right)^a$ which give

$$q_z = \frac{\left(\frac{q}{p}\right)^z + \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^a} \qquad (p \neq q)$$

If $p = q$ we have a repeated root $\lambda = 1$, so we have the general solution $(A + Bz) \cdot 1^z$ and a particular solution, applying the boundary conditions, $q_z = 1 - \dfrac{z}{a}$.

To find the expected duration of the game

$$\begin{aligned} E_z &= p(1 + E_{z+1}) + q(1 + E_{z-1}) + r(1 + E_z) \\ E_z &= \frac{1}{p+q} + \frac{p}{p+q}E_{z+1} + \frac{q}{p+q}E_{z-1} \end{aligned}$$

with boundary conditions $E_0 = 0$ and $E_a = 0$.
The general solution of the homogeneous equation is

$$\begin{aligned} A + B\left(\frac{q}{p}\right)^z &\qquad p \neq q \\ A' + B'z &\qquad p = q \end{aligned}$$

Particular solutions to try for the non-homogeneous equation are $E_z = Cz$ for $p \neq q$ or $E_z = Dz^2$ if $p = q$
Applying the boundary conditions gives:

$$\begin{aligned} E_z &= \frac{z}{q-p} - \frac{a}{q-p}\left(\frac{1 - \left(\frac{q}{p}\right)^z}{1 - \left(\frac{q}{p}\right)^a}\right) &\qquad p \neq q \\ E_z &= \frac{z(a-z)}{p+q} &\qquad p = q \end{aligned}$$