Question

In the Gambler's ruin problem, changing the stakes from 1 unit to $\frac{1}{2}$ unit is equivalent to doubling the initial capitals. What effect does this have on the game?

Answer

In the gambler's ruin problem, to investigate the effect of doubling the initial capital.

(i) $p = q: \quad q_z = 1 - \frac{z}{a}$

So if z is replaced by $2z$ and a by $2a$ q_z is unchanged.

$E_z = z(a - z)$, so if $z \to 2z$ and $a \to 2a$

$E_{2z} = 2z(2a - 2z) = 4E_z$

(ii) $p \neq q: \quad q_z = \dfrac{\left(\frac{q}{p}\right)^z - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^a}$

$$E_z = \frac{z}{q-p} - \frac{a}{q-p}\left(\frac{1 - \left(\frac{q}{p}\right)^z}{1 - \left(\frac{q}{p}\right)^a}\right)$$

Replacing z by $2z$ and a by $2a$

$$\begin{aligned} q_z \quad &\to \quad \frac{\left(\frac{q}{p}\right)^{2z} - \left(\frac{q}{p}\right)^{2a}}{1 - \left(\frac{q}{p}\right)^{2a}} = \frac{\left(\frac{q}{p}\right)^z - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^a} \cdot \frac{\left(\frac{q}{p}\right)^z + \left(\frac{q}{p}\right)^a}{1 + \left(\frac{q}{p}\right)^a} \\ &= \quad q_z \cdot K \end{aligned}$$

Where $K > 1$ if $q > p$ and $K < 1$ if $q < p$.

or:

$$q_z \to \frac{\left[\left(\frac{q}{p}\right)^2\right]^z - \left[\left(\frac{q}{p}\right)^2\right]^a}{1 - \left[\left(\frac{q}{p}\right)^2\right]^a}$$

So the game is "equivalent "to the original with the odds changed from $p : q$ to $p^2 : q^2$ with draws allowed since $p^2 + q^2 < 1$

$$E_z \to \frac{2z}{q-p} - \frac{2a}{q-p}\left(\frac{1 - \left(\frac{q}{p}\right)^{2z}}{1 - \left(\frac{q}{p}\right)^{2a}}\right)$$