Question

If $N_1(t)$ and $N_2(t)$, $t \geq 0$, are two independent Poisson processes with rates λ_1 and λ_2 respectively, obtain an expression for

$$P\left(N_1(t) = n_1 | N_1(t) + N_2(t) = n\right),$$

where $n_1 \leq n$, and comment on your result.

Answer

$P(N_1(t) = n_1 | N_1(t) + N_2(t) = n)$

$$\begin{aligned}
&= \frac{P(N_1(t) = n_1 \text{ and } N_2(t) = n - n_1)}{P(N_1(t) + N_2(t) = n)} \\
&= \frac{P(N_1(t) = n_1) \times P(N_2(t) = n - n_1)}{P(N_1(t) + N_2(t) = n)} \\
&\quad (N_1,\ N_2 \text{ are independent}) \\
&= \frac{\frac{e^{-\lambda_1 t}(\lambda_1 t)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-n_1}}{(n-n_1)!}}{\frac{e^{-(\lambda_1+\lambda_2)t}(\lambda_1+\lambda_2)^n t^n}{n!}} \\
&\quad (N_1 + N_2 \text{ is Poisson with rate } \lambda_1 + \lambda_2) \\
&= \frac{n!}{n_1!(n-n_1)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n_1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-n_1}
\end{aligned}$$

This is the binomial probability of n_1 successes in n trials, the probability of success being $\dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

A Bernoulli trial is an event in the combined process, and a success occurs if the event is from the first process N_1.