Question

Suppose events occur in a Possion process with rate λ.

(i) Find the conditional probability that there are m events in the first s units of time, given that there are n events in the first t units of time, where $0 \leq m \leq n$ and $0 \leq s \leq t$.

(ii) If $N(t)$ denotes the number of events in $(0, t]$ and T denotes the time until the first event, find $P\{T \leq s | N(t) = n\}$ for $0 \leq s \leq t$ and n a positive integer.

Answer

(i) $P(N(s) = m | N(t) = n)$

$$= \frac{P(N(s) = m \,\text{and}\, N(t) = n)}{P(N(t) = n)}$$

$$= \frac{P(N(0, s) = m) \times P(N(s, t) = n - m)}{P(N(t) = n)}$$

since no. of events in $(0, s]$ and $(s, t]$ are independent.

$$= \frac{P(N(s) = m) \times P(N(t - s) = n - m)}{P(N(t) = n)}$$

$$= \frac{e^{-\lambda s}(\lambda s)^m}{m!} \times \frac{e^{-\lambda(t-s)}(\lambda(t - s))^{n-m}}{(n - m)!} \times \frac{n!}{e^{-\lambda t}(\lambda t)^n}$$

$$= \frac{n!}{m!(n - m)!} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}$$

(note that this is a binomial probability)

(ii) $T \leq s$ is equivalent to $N(s) \geq 1$

$$P(T \leq s | N(t) = n) = \sum_{m=1}^{n} P(N(s) = m | N(t) = n)$$

$$= \sum_{m=1}^{n} \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m} \text{ by (i)}$$

$$= 1 - \left(1 - \frac{s}{t}\right)^n$$