Question

Customers arrive for a service taking one hour and provided by only one server. If there are any customers waiting for service at the beginning of an hour, exactly one person will be served during that hour. If no customers are waiting for service at the beginning of an hour, none will be served during the hour. Suppose that Z_n new customers arrive during the n-th period, where $Z_1, Z_2, ...$ are independent, identically distributed random variables with distribution $P(Z = k) = c_k,\ (k = 1, 2, \ldots)$. Let X_n denote the number of customer present at the end of the n-th period $(n \geq 1)$ and let X_0 be the initial number of customers. Check that $\{X_n\} \quad (n = 0, 1, 2, \ldots)$ is a Markov chain and show that its transition probability matrix has the form

$$\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \cdots \\ c_0 & c_1 & c_2 & c_3 & \cdots \\ 0 & c_0 & c_1 & c_2 & \cdots \\ 0 & 0 & c_0 & c_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Answer

Let X_n denote the number of customers waiting for service at the end of the n-th hour, and let Z_n denote the number of customers who arrive to be served during the n-th hour.

If $X_n = 0$ then $X_{n+1} = Z_{n+1}$

If $X_n > 0$ then $X_{n+1} = X_n - 1 + Z_{n+1}$

$P(X_{n+1} = k | X_n = 0) = P(Z_{n+1} = k) = c_k$

and so this doesn't depend on previous history, but only on how many customers arrive during the n-th hour.

$P(X_{n+1} = k | X_n = j) = P(Z_{n+1} = k - j + 1) = c_{k-j+1}$ for $k \geq j - 1$ and $j \geq 1$

$P(X_{n+1} = k | X_n = j) = 0$ for $k < j - 1$

Again these depend only on the number of customers arriving during the $(n+1)$-th period and the number at the beginning of the period, and not on previous history.

so

$$P \quad = \quad \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \cdots \\ c_0 & c_1 & c_2 & c_3 & \cdots \\ 0 & c_0 & c_1 & c_2 & \cdots \\ 0 & 0 & c_0 & c_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$