Question

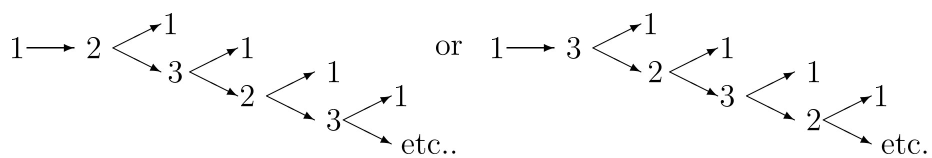
Classify as transient, null-recurrent or positive recurrent the states of the Markov chains with the following transition probability matrices:

(a) $$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

(b) $$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Answer

(i) Clearly from symmetry each state is of the same type. Return to state 1 may be achieved by two sets of routes:



So $f_{11} = 2\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots\right) = 1$

The mean recurrence time is given by

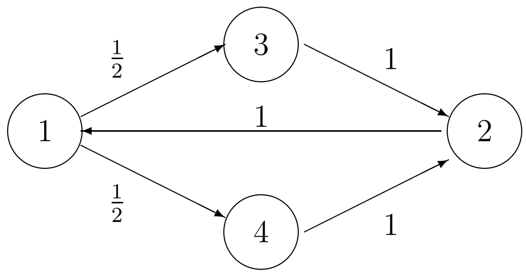
$$\begin{aligned} \mu_1 &= 1\cdot 0 + 2\cdot 2\cdot\left(\frac{1}{2}\right)^2 + 3\cdot 2\cdot\left(\frac{1}{2}\right)^3 + 4\cdot 2\cdot\left(\frac{1}{2}\right)^4 + \ldots \\ &= 2\cdot\frac{1}{2} + 3\cdot\left(\frac{1}{2}\right)^2 + 4\cdot\left(\frac{1}{2}\right)^3 + \ldots \end{aligned}$$

Note that $\frac{1}{2}\mu_1 = 2\cdot\left(\frac{1}{2}\right)^2 + 3\cdot\left(\frac{1}{2}\right)^3 + \ldots$

so $\mu_1 - \frac{1}{2}\mu_1 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \frac{3}{2}$ so $\mu_1 = 3$

Thus state 1 is positive recurrent and similarly states 2 and 3 are positive recurrent with $\mu = 3$

(ii) Transition diagram



There are two return routes $1 \to 1$, namely $1 \to 3 \to 2 \to 1$ and $1 \to 4 \to 2 \to 1$.

So $f_{11} = \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 1 = 1$

The recurrence time for each rout is 3 so $\mu_1 = 3$

So $f_{22} = \underbrace{1 \cdot \frac{1}{2} \cdot 1}_{2-1-3-2} \; + \; \underbrace{1 \cdot \frac{1}{2} \cdot 1}_{2-1-4-2} = 1 \qquad \text{Again } \mu_2 = 3$

So $f_{33} = \underbrace{1 \cdot 1 \cdot \frac{1}{2}}_{3-2-1-3} \; + \; \underbrace{1 \cdot 1 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{1}{2}}_{3-2-1-4-2-1-3} \; + \; \underbrace{\frac{1}{2}^3 + \frac{1}{2}^4 + \ldots}_{1-4-2 \text{ cycle}} = 1$

$\mu_3 = 3 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2}^2 + 9 \cdot \frac{1}{2}^3 + \ldots 3 \left[\frac{1}{2} + 2 \cdot \frac{1}{2}^2 + 3 \cdot \frac{1}{2}^3 + \ldots \right] = 6$

By symmetry $f_{44} = 1$ and $\mu_4 = 6$