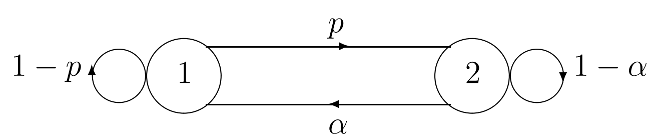
Question

A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to nonfilter cigarettes the next week with probability p. On the other hand, if he smokes nonfilter cigarettes one week there is a probability α that he switches to filter in the following week. Classify the states of this Markov chain for different values of p and α as transient, null-recurrent or positive recurrent.

Answer

This is a 2-state Markov chain with state 1 "smokes filter "and state 2 "smokes non-filter ".

The transition matrix is $P = \begin{pmatrix} 1-p & p \\ \alpha & 1-\alpha \end{pmatrix}$



If $p = 0$ state 1 is absorbing
If $\alpha = 0$ state 2 is absorbing
If $p = 0$ and $\alpha > 0$ state 2 is transient
If $\alpha = 0$ and $p > 0$ state 1 is transient
Suppose $p \neq 0$ and $\alpha \neq 0$
P(returning to state 1) $= f_{11}$
$= (i - p) + p\alpha + p(1-\alpha)\alpha + p(1-\alpha)^2\alpha + ...$
$= (1 - p + p\alpha(1 + (1-\alpha) + (1-\alpha)^2 + ...)$
$= 1 - p + \dfrac{p\alpha}{1-(1-\alpha)} = 1$
so state 1 is recurrent. Similarly state 2 is recurrent.

Mean recurrence time for state 1:

$$\begin{aligned} \mu_1 &= 1 \cdot (1-p) + 2pa + 3p(1-\alpha)\alpha + 4p(1-\alpha)^2\alpha + ... \\ &= 1 + \frac{p}{\alpha} \end{aligned}$$

using arithmetic - geometric series.
Similarly $\mu_2 = 1 + \dfrac{\alpha}{p}$