Question

Consider the Markov chain having state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Find P^2

(b) Show that $P^4 = P^2$

(c) Find P^n, $n \leq 1$.

If the system starts (at step 0) in state 1, find the probability that it occupies the different states at step n.

Answer

$$\begin{aligned} P &= \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \end{pmatrix} \\ P^2 &= \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \\ P^3 &= \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \end{pmatrix} = P \end{aligned}$$

So $P^2 = P^4$

Also $P^n = P$ if n is odd and $P^n = P^2$ if n is even.

So if $\mathbf{p}_0 = (0, 1, 0)$, $\quad \mathbf{p}^{(n)} = \mathbf{p}_0 P^n = \begin{cases} (\frac{1}{4}, 0, \frac{3}{4}) & n \text{ odd} \\ (0, 1, 0) & n \text{ even} \end{cases}$

So after an even number of steps the system occupies state 1. After an odd number of steps it occupies state 0 with probability $\frac{1}{4}$ and state 2 with probability $\frac{3}{4}$.