Question

In a three-dimensional walk a particle starts at an origin in three-space which can be thought of as the centre of a $2 \times 2 \times 2$ cube. At each step the particle moves north or south (each with probability $\frac{1}{2}$), east or west (each with probability $\frac{1}{2}$) and up or down (each with probability $\frac{1}{2}$). Thus in one step the particle moves to one of the eight corners of the cube. If the walk continues forever show that the particle is not certain to return to the origin.

Answer

The reasoning is very similar to the 1-dimensional case. To return to 0 the particle must do so as a result of a simultaneous return in in respect of movement all three directions.

$$P(\text{Particle is at 0 after } k \text{ steps}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^3 & \text{if } k = 2n \end{cases}$$

The expected number of returns to the origin is

$$\begin{aligned} E &= \sum_{k=1}^{\infty} E(R_k) \cdot R_k = \begin{cases} 1 & \text{if particle at 0 after } k \text{ steps} \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{k=1}^{\infty} P(R_k = 1) \\ &= \sum_{n=1}^{\infty} \left[\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]^3 \\ &\approx \sum_{n=1}^{\infty} \left(\frac{1}{n\pi}\right)^{\frac{3}{2}} \end{aligned}$$

using Stirling's approximation. The series converges, so $E < \infty$. Standard theory gives $P = 1 - \dfrac{1}{1+E}$ so $P < 1$